Mathematical

Reviews

Edited by

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Mathematical Reviews

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Haupt, Otto. Über die kombinatorische Mindestordnung signierter Permutationen. J. Reine Angew. Math. 186, 221–229 (1949).

It is shown that the (combinatorial) least order of a signed permutation of n elements is $n+L$, L being the length defined earlier [see Denk and Haupt, same vol., 170–183 (1945); these Rev. 10, 670]. This replaces an earlier rule [Denk and Haupt, same J. 183, 69–91 (1941); these Rev. 4, 113], where the concept of least order is also defined.

J. Riordan (New York, N. Y.).

Gardedieu, A. Sur quelques formules d'analyse combinatoire. Mathesis 57, 83–102 (1948).

The combinatorial formulas are (i) the relation of binomial coefficients implicit in the identity $\prod(1+x)^{a_i}=(1+x)^a$, $a=\sum a_i$, and (ii) the number of terms, for any power of x, in the sum corresponding to the left hand side. Since the terms corresponding to $(1+x)^{a_i}$ are enumerated by $1+x+\cdots+x^{a_i}$, the second formula is for the number of combinations of objects, a_i of which are of the ith kind, or for the numbers enumerated by $(1-x)^{-n}\prod(1-x^{a_i+1})$, when i runs from 1 to n, a result which would have compressed the author's piecemeal progression. *J. Riordan.*

Ryser, H. J. A note on a combinatorial problem. Proc. Amer. Math. Soc. 1, 422–424 (1950).

The author proves the following theorem: If v distinct elements $x_1, \cdots, x_v$ are arranged into v sets, each of k elements, such that 2 of these sets have exactly λ elements in common, then $\lambda(v-1)=k(k-1)$. The author's proof uses the incidence matrix of the configuration. The author remarks that an alternative proof can be given using an argument of R. C. Bose [Ann. Eugenics 9, 353–399 (1939); these Rev. 1, 199]. *H. B. Mann* (Columbus, Ohio).

Frame, J. S. Note on the product of power sums. Pi Mu Epsilon J. 1, 18–21, errata p. 48 (1949).

Define $\zeta_p(n)=1^p+\cdots+n^p$. The author obtains formulas for $\zeta_p(n)\cdot\zeta_q(n)$. *P. Erdős* (Aberdeen).

Throumoulopoulos, L. On the impossibility of the identity $X^\mu+Y^\nu=1$. Bull. Soc. Math. Grèce 24, 73–75 (1949). (Greek)

By means of an elementary argument concerning degrees, the author proves the following theorem. If $\pi(z)$ and $\varphi(z)$ are polynomials, then an identity of the form $\pi^\mu(z)+\varphi^\nu(z)=1$ is impossible if μ and ν are positive integers subject to the condition $1/\mu+1/\nu\leqq1$. *T. M. Apostol.*

Parodi, Maurice. Sur une méthode de formation de matrices définies positives. Ann. Soc. Sci. Bruxelles. Sér. I. 63, 127–129 (1949).

A method for the construction of families of positive definite matrices of any order is presented. The procedure is based on a theorem of Ostrowski [Bull. Sci. Math. (2) 61, 19–32 (1937), p. 30], by means of which it is possible to find an upper bound d_n which may be subtracted or added to each element of a matrix without changing the sign of the value of its determinant. *H. Polachek* (White Oak, Md.).

Gyires, B. Über vertauschbare Matrizen. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 143–145 (1950).

Proofs of two theorems of Frobenius [J. Reine Angew. Math. 84, 1–63 (1877)]. *N. H. McCoy.*

Souriau, J. M. Les calculs matriciel & spinoriel. O.N.E.R.A. Publ. no. 42, vi+27 pp. (1950).

Lee, H. C. Eigenvalues and canonical forms of matrices with quaternion coefficients. Proc. Roy. Irish Acad. Sect. A. 52, 253–260 (1949).

Let the numbers of the real quaternion domain Q be written in the form $a=a_1+ja_2$ with a_1 and a_2 in a commutative subfield C isomorphic with the complex numbers, $j^2=-1$ and $cj=j\bar{c}$ for c and $\bar{c}$ in C. A quaternion matrix A is said to have eigenvalue t, which is always required to belong to C, if $Av=vt$ for v a nonzero column vector with quaternion elements. [There is no discussion of the lack of invariance in the choice of C. The results of the paper are clarified by the remark that if τ is an eigenvalue associated with v but not required to be in C, then $\sigma^{-1}\tau\sigma$ is the eigenvalue of $v\sigma$ for arbitrary nonzero σ in Q. A conjugate class $\sigma^{-1}\tau\sigma$ contains just one or two elements of C (t and $\bar{t}$), which may be taken as canonical representatives of the "eigenclass" $\sigma^{-1}\tau\sigma$.] Using as a tool the isomorphism

$$A=A_1+jA_2\leftrightarrow\begin{pmatrix}A_1 & -\bar{A}_2\\ A_2 & \bar{A}_1\end{pmatrix}=f(A)$$

between quaternion matrices of order n and a certain ring of matrices of order $2n$ over C, the following results are obtained: An n by n quaternion matrix A has $2n$ (complex) eigenvalues which occur in conjugate pairs and which are the eigenvalues (in the ordinary sense) of $f(A)$; A is similar by means of a unitary quaternion matrix to a triangular matrix with the (complex) diagonal elements unique to within order and individual replacement by their conjugate; and, two normal matrices are unitarily equivalent if and only if they have the same $2n$ eigenvalues, their canonical form under unitary similarity being diagonal. On p. 257 the displayed matrix T_2 should have diagonal elements zero and there are some other easily corrected misprints. *W. Givens* (Knoxville, Tenn.).

van der Mey, G. Sylvester's determinant. Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. 19, no. 3, 39 pp. (1949). (Dutch)

Let $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ be polynomials of degree n and m of the homogeneous variables x_1, x_2, x_3. The author considers them as polynomials of x_1 only and defines as

$R(f, g)$ their determinant of Sylvester; R is a polynomial of degree nm of the homogeneous variables x_2, x_3. The starting point of his investigations on the multiplicity of the common points of the curves $f = 0$ and $g = 0$ is R. He gives, moreover, a discussion of the multiple points of a curve and a proof of Plücker's formulae.   *O. Bottema* (Delft).

Glenn, Oliver E. Inverse processes in invariants, with applications to three problems in mechanics. Ann. Scuola Norm. Super. Pisa (2) **15** (1946), 39–95 (1950).

The author considers what are called invariant problems in invariant theory. Let an invariant $D(x_1, x_2, a_i)$ be given together with the way in which the a_i transform under transformation of x_1 and x_2. Then the ground-quantic $f(x_1, x_2, a_i)$, linear in a_i, and the transformation group of x_1 and x_2 are to be determined. It is shown that this problem is solvable in special cases.   *J. Haantjes* (Leiden).

Jarušek, Jaroslav. Semi-invariants of ternary forms. Rozpravy II. Třídy České Akad. **56**, no. 1, 40 pp. (1946). (Czech)

The author investigates the classical theorem of finding a complete system of concomitants for a pair of ternary quadratics under the full linear group by means of a systematic procedure, wherein one variable is segregated and each of the quadratics is regarded as a linear aggregate of binary forms, namely, of a quadratic or linear form and a constant in the other two variables. This reduces the problem in ternary forms to one in binary forms, but only by increasing the complexity of the ground forms under consideration. The technique is by means of six operators $\Delta_{12}, \Delta_{21}, \Delta_{13}, \Delta_{31}, \Delta_{23}, \Delta_{32}$ which, acting on a coefficient a_{ij} of a ternary form $\sum_{i,j} \binom{n}{h, i, j} a_{ij} x_1{}^h x_2{}^i x_3{}^j$ where $h + i + j = n$, a constant positive integer, raise or lower a single suffix, or both suffixes, by a unit step. The operators apply also to contravariant forms (in contragredient variables u_i) and to mixed forms. The method applies to nonsymbolic and to symbolic notation of the expressions. While being considerably longer than the classical symbolic treatment of Clebsch and Gordan the method is none the less systematic, and it leads to the syzygies pertaining to the complete system; and also to the complete system of a single ternary bilinear form.   *H. W. Turnbull* (St. Andrews).

Abstract Algebra

Kurepa, Georges. Sur la notion de processus (fonction générale). C. R. Acad. Sci. Paris **231**, 316–318 (1950).

The author considers a partial ordering by "extension" of general "processes," or (not-necessarily single-valued) functions. His definitions are closely connected with the usual algebra of relations. Applications to the Souslin problem will be given elsewhere.   *G. Birkhoff.*

Hermes, Hans. Grundbegriffe der Verbandstheorie. Schr. Math. Inst. Univ. Münster, no. **2**, 20 pp. (undated).

Exposition of basic definitions, theorems, and examples of lattice theory, including modular and distributive lattices, Boolean algebras, and lattices of subgroups.   *P. M. Whitman* (Silver Spring, Md.).

Klein-Barmen, Fritz. Zur Axiomatik der ausgeglichenen Gewebe. Math. Z. **53**, 70–75 (1950).

The author shows that various elementary properties of semi-modular lattices are valid in any partially ordered set of finite length in which, if a and b cover a common element, they are also covered by a common element.
                    G. Birkhoff (Cambridge, Mass.).

Ellis, David. An algebraic characterization of lattices among semilattices. Portugaliae Math. **8**, 103–106 (1949).

It is noted that a lattice may be defined as a semilattice in which any two elements have least common multiple. [See also R. P. Dilworth and M. Ward, Bull. Amer. Math. Soc. **55**, 1141 (1949); these Rev. **11**, 309.]
                    G. Birkhoff (Cambridge, Mass.).

Harary, Frank. Atomic Boolean-like rings with finite radical. Duke Math. J. **17**, 273–276 (1950).

The author discusses the structure of the "Boolean-like" rings introduced by A. L. Foster [Trans. Amer. Math. Soc. **59**, 166–187 (1946); these Rev. **7**, 360]. He gives a complete classification of the isomorphism-types of finite Boolean-like rings; some preliminary results hold for atomic Boolean-like rings with finite radical. The basic new concept is that of a "simple" nonzero element of the radical of an atomic Boolean-like ring, or element z satisfying $az = z$ for some atom a.

                    G. Birkhoff (Cambridge, Mass.).

Lyapin, E. S. Semisimple commutative associative systems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 367–380 (1950). (Russian)

A commutative associative system (herein abbreviated "system") is called semisimple provided it possesses no normal complexes other than ideals and single elements. Compare terminology in reviews of earlier papers [same vol., 179–192, 275–282 (1950); these Rev. **11**, 575; **12**, 5]. A system is semisimple if and only if it possesses no homomorphisms other than isomorphisms or ideal-homomorphisms. The latter is a mapping, as described by Rees, upon a difference semigroup modulo an ideal [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127]. If $\mathfrak{A}$ and $\mathfrak{B}$ are systems, $\mathfrak{A} = \mathfrak{B} \cup E$, where E is the identity element of $\mathfrak{A}$ and $E \bar\in \mathfrak{B}$, then $\mathfrak{A}$ is obtainable from $\mathfrak{B}$ by exterior adjunction of an identity. A semisimple system with an identity either is a cyclic group with order one or a prime, or is of order two, or is obtainable by exterior adjunction of an identity to a semisimple system without an identity; the converse is false. Whenever elements A, B in a system $\mathfrak{A}$ satisfy $A = AB$, then B is called a partial identity for A. A semisimple system is decomposable, and uniquely so, as a mutually-annihilating sum of indecomposable semisimple systems. The four principal theorems classify all semisimple systems (other than groups) by giving necessary and sufficient conditions that a system belong to one of the following four types: (1) semisimple, containing an identity; (2) semisimple, containing no identity, indecomposable; (3) semisimple, decomposable, every element possessing a partial identity; (4) semisimple, decomposable, at least one element possessing no partial identity.

                    R. A. Good (College Park, Md.).

Herglotz, Gustav. Eine Formel der formalen Operatoren-rechnung. Math. Ann. **122**, 14–15 (1950).

The author shows that if $XY - YX = Z$, where Z is commutative with both X and Y, then

$$e^{tXY} = \sum_0^\infty (X^\nu Y^\nu / \nu!)[(e^{tZ} - 1)/Z]^\nu.$$

The formula has been previously established by the reviewer [Proc. Edinburgh Math. Soc. (2) **3**, 118–127 (1932)].

N. H. *McCoy* (Northampton, Mass.).

Rédei, L., und Szele, T. Die Ringe "ersten Ranges." Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 18–29 (1950).

An Abelian group G has rank 1 whenever any two cyclic subgroups, not 0, have intersection, not 0; and G is locally cyclic whenever any two of its elements generate a cyclic subgroup. The authors give a derivation of the well-known classification of these groups; and they give a survey of all the rings with locally cyclic additive groups. Apart from trivial rings these are just the well-known subrings of the field of rational numbers and the direct sums of certain trivial rings and finite rings derived from cyclic groups of prime power order.

R. *Baer* (Urbana, Ill.).

Krull, Wolfgang. Subdirekte Summendarstellungen von Integritätsbereichen. Math. Z. **52**, 810–826 (1950).

An integral domain is understood to be a proper integral domain, that is, not a field. A subdirect sum of a finite number of fields cannot be an integral domain. If $\Re_\alpha$ is a set of any number of fields such that the number of elements in each field does not exceed a fixed bound N, then no subdirect sum of the fields $\Re_\alpha$ can be an integral domain. If M is a set of fields, let us denote by M^n the set of fields in M which have characteristic n, n either zero or a prime. No subdirect sum of the fields in M can be an integral domain if the following three conditions are satisfied: (a) The fields in M do not all have the same characteristic; (b) M^0 contains only a finite number of fields; (c) there exists a finite number of primes $p_1, \cdots, p_k$ such that the number of elements of each field in $M - (M^0 \cup M^{p_1} \cup \cdots \cup M^{p_k})$ does not exceed a fixed bound N. Now let M be a countable set of finite or at most countably infinite fields $\Re_i$ such that the three stated conditions are not all satisfied. If $\Im$ is an integral domain which can be represented as a subdirect sum of the fields $\Re_i$, then each polynomial ring $\Im[u_1, \cdots, u_n]$ in a finite number of indeterminates is so representable, as is also the polynomial ring $\Im[u_1, u_2, \cdots]$ in a countably infinite number of indeterminates. If $\Re_i$ is a countable set of finite or at most countably infinite fields which satisfies condition (a) but not both (b) and (c), then the ring $\Im_0[u_1, u_2, \cdots]$ of all polynomials in a countably infinite number of indeterminates with rational integral coefficients is a subdirect sum of the $\Re_i$. Corresponding results are obtained if the fields $\Re_i$ all have the same characteristic. If α is an ideal in an integral domain $\Im$ which can be represented as a subdirect sum of fields, the components of α in the representation do not determine the arithmetical properties of α. The author briefly discusses special subdirect sums as studied by Köthe [Math. Ann. **103**, 545–572 (1930)], but using the terminology of the reviewer [Bull. Amer. Math. Soc. **53**, 856–877 (1947); these Rev. 9, 77].

N. H. *McCoy* (Northampton, Mass.).

Amitsur, A. S., and Levitzki, J. Minimal identities for algebras. Proc. Amer. Math. Soc. **1**, 449–463 (1950).

Let A be the n by n total matrix algebra over a field F. Then A satisfies the identity $\sum \pm x_1 x_2 \cdots x_{2n} = 0$, the sum being taken over all permutations, with the sign prefixed according to the parity. This theorem, which had been conjectured for some time, is proved by a careful induction on the number of idempotents among the x's. It is further shown that there are no identities of degree $< 2n$, and that those of degree $2n$ are essentially of this form, except when $n \leq 2$ and F has 2 elements. In the final section of the paper, the results are extended to semisimple algebras.

I. *Kaplansky* (Chicago, Ill.).

Theory of Groups

Pták, Vlastimil. Immersibility of semigroups. Acta Fac. Nat. Univ. Carol., Prague no. **192** (1949), 16 pp. (1949).

A semi-group S is taken to be an associative multiplicative system in which $ax = ax'$ and $xb = x'b$ each implies $x = x'$. It is shown that if S contains two elements a and j such that $aj = a$ (or $ja = a$), then j is the unit of S, and hence that a unit element can be adjoined to S if it possesses none. It is then assumed that S has a unit element. Let Γ be a set of generators of S. The free semi-group on Γ is defined to be the set S^* of elements $\prod_{i=1}^n a_i$, where the a_i are in Γ. Then $S^* \sim S$ under a homomorphism $H: H(\alpha) \epsilon S$ where $\alpha \epsilon S^*$. The free group G^* on Γ is the set of elements $\prod_{i=1}^n a_i^{\epsilon_i}$ where $\epsilon_i = \pm 1$. The semi-group S is contained in a group G with generators Γ if and only if there is a homomorphism of G^* on G which preserves the homomorphism H of S^* on S. Let $\mathfrak{M}_r$ be the set of elements of G^* of the form $\alpha\beta^{-1}$ where α and β are in S^* and $H(\alpha) = H(\beta)$. Let $\{\mathfrak{M}_r\}$ be the intersection of all subgroups of G^* which contain $\mathfrak{M}_r$. It can be shown that, if α and β are in S^*, then $\alpha\beta^{-1}$ in $\{\mathfrak{M}_r\}$ implies $\alpha\beta^{-1}$ in $\mathfrak{M}_r$. If $[\mathfrak{M}_r]$ is the intersection of all normal subgroups of G^* which contain $\mathfrak{M}_r$, then S is immersible in a group if and only if $\alpha\beta^{-1}$ in $[\mathfrak{M}_r]$ implies $\alpha\beta^{-1}$ in $\mathfrak{M}_r$. Hence, if $\{\mathfrak{M}_r\} = [\mathfrak{M}_r]$, S is immersible. A semi-group S with unit, such that a, b in S implies that $ua = vb$ for some elements u, v in S, is called an Ore semi-group. It is shown that an Ore semi-group satisfies the condition that $\alpha\beta^{-1}$ in $[\mathfrak{M}_r]$ implies $\alpha\beta^{-1}$ in $\mathfrak{M}_r$, and is therefore immersible [see O. Ore, Ann. of Math. (2) **32**, 463–477 (1931)]. The methods employed appear to be closely related to the work of Malcev [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 331–336 (1939); these Rev. 2, 7].

F. *Kiokemeister.*

Murata, Kentaro. On the quotient semi-group of a non-commutative semi-group. Osaka Math. J. **2**, 1–5 (1950).

The results of K. Asano [J. Math. Soc. Japan **1**, 73–78 (1949); these Rev. 11, 154] are shown to apply to a semi-group.

R. E. *Johnson* (Northampton, Mass.).

Mattioli, Ennio. Sopra una particolare proprietà dei gruppi abeliani finiti. Ann. Scuola Norm. Super. Pisa (3) **3** (1949), 59–65 (1950).

Let (*) $n = (p^k - 1)(p - 1)^{-1}$, where p is prime and $k \geq 2$. Then if G is a commutative group with generators $R_1, \cdots, R_n$, each of order p, there is a subgroup Γ such that

$$G = \Gamma + \sum_{i=1}^n \sum_{j=1}^{p-1} \Gamma R_i^j.$$

If (*) is true, then from the p^n possible permutations n at a time (with repetition) of p distinct elements, p^{n-k} may be so chosen that each of the remaining permutations differs from at least one of them by at most one element. If the elements are "win," "lose," and "draw," it is possible to make 3^{n-2} forecasts of the result of n football matches and be certain of having one with at most one error, if $n = \frac{1}{2}(3^k-1)$ where $k \geq 2$. If (*) is true and $p=2$, there are $_nC_2/3$ trios of n given elements such that each pair of elements is contained in just one of the trios.   *H. A. Thurston.*

Hirsch, K. A. On a theorem of Burnside. Quart. J. Math., Oxford Ser. (2) 1, 97–99 (1950).

Let G be a group of order N, let d be the greatest common divisor of all the numbers p^2-1, where the p's are prime divisors of N, and r the number of conjugate sets in G. Then (1) $N \equiv r \pmod{2d}$ for N odd; and (2) $N \equiv r \pmod 3$ for N even and prime to 3. The number of pairs of commuting elements X and Y, not both 1, is $Nr-1$. Writing the group $\{X, Y\}$ as a direct product of Sylow-groups, and counting modulo d the number of pairs of generators X_i and Y_i for each factor, shows that $Nr-1 \equiv 0 \pmod d$. But $N^2 \equiv 1 \pmod d$, whence (3) $N \equiv r \pmod d$, a result containing (2). The stronger congruence (1) is obtained, using (3), by a similar count modulo $2^{\beta+1}$, where 2^β is the highest power of 2 that divides d. It is noted that the modulus $2d$ in (1) is the best possible.   *R. C. Lyndon* (Princeton, N. J.).

Čunihin, S. A. On Sylow properties of finite groups. Doklady Akad. Nauk SSSR (N.S.) 73, 29–32 (1950). (Russian)

Let $\mathfrak{G}$, g, Π, m be as described in a review of an earlier paper by the author [same Doklady (N.S.) 66, 165–168 (1949); these Rev. 10, 678]. The group $\mathfrak{G}$ is said to be of type $\Pi-2$ provided $\mathfrak{G}$ contains a subgroup $\mathfrak{A}$ of order $a=g/m$ and a solvable subgroup $\mathfrak{B}$ of order m such that every subgroup of $\mathfrak{G}$ of order a is conjugate to $\mathfrak{A}$ and every subgroup of $\mathfrak{G}$ of order m is conjugate to $\mathfrak{B}$. A type $\Pi-2$ group is always of type $\Pi-1$. A finite group $\mathfrak{G}$ is of type $\Pi-2$ if and only if $\mathfrak{G}$ has a normal series, every factor-group of which is of type $\Pi-2$. Every Π-solvable group is of type $\Pi-2$. If $\mathfrak{G}$ is Π-solvable, every subgroup whose order divides either m or a is contained in a subgroup whose order is, respectively, m or a.   *R. A. Good.*

Plotkin, V. I. On the theory of noncommutative groups without torsion. Doklady Akad. Nauk SSSR (N.S.) 73, 655–657 (1950). (Russian)

An R-group, of which every factor-group modulo an invariant isolated subgroup is also an R-group, is called an R^*-group. For terminology see the review of a paper by Kontorovič [Mat. Sbornik N.S. 22(64), 79–100 (1948); these Rev. 9, 493]. A normal series, every factor of which is isomorphic to a subgroup of the additive group of rational numbers is called a rational series. Let $\mathfrak{G}$ be an R^*-group with an ascending invariant rational series. Then $\mathfrak{G}$ contains an isolated invariant subgroup $\mathfrak{G}'$ such that $\mathfrak{G}'$ has an ascending central series and the quotient group $\mathfrak{G}/\mathfrak{G}'$ is Abelian and torsion-free. Through an arbitrary isolated invariant subgroup of $\mathfrak{G}$ can be passed an ascending invariant rational series. Additional results are found for a particular case of R^*-groups, namely torsion-free groups with the normalizer condition. In such a group, the existence of an ascending invariant rational series is equivalent to the existence of an ascending central series.   *R. A. Good.*

Černikov, S. N. Periodic ZA-extensions of complete groups. Mat. Sbornik N.S. 27(69), 117–128 (1950). (Russian)

A group possessing an ascending central series is called a ZA-group. Any extension of one group by another is called a ZA-extension provided the extension is a ZA-group. The extension of an element A in a group is the subgroup generated by the solutions of all possible equations $X^n = A^m$, where m and n are integers and $n \neq 0$. Let $\mathfrak{H}$ denote a torsion-free ZA-group. Then $\mathfrak{H}$ is an R-group, in the sense of Kontorovič [Mat. Sbornik N.S. 22(64), 79–100 (1948); these Rev. 9, 493]. Every factor of the upper central series of $\mathfrak{H}$ is torsion-free. A normal subgroup $\mathfrak{N}$ of $\mathfrak{H}$ is a servant subgroup if and only if $\mathfrak{H}/\mathfrak{N}$ is torsion-free. If $\mathfrak{H}$ contains a normal subgroup $\mathfrak{N}$ isomorphic to a subgroup of the additive group of rational numbers, then $\mathfrak{N}$ is contained in the center of $\mathfrak{H}$. In $\mathfrak{H}$ the extension of every element $A \neq 1$ is an Abelian group of rank one and maximal among such subgroups containing A. The extensions of any two of the elements of $\mathfrak{H}$ either coincide or intersect in the identity alone.

If a ZA-group $\mathfrak{G}$ contains a torsion-free, servant, normal subgroup $\mathfrak{A}$ such that $\mathfrak{G}/\mathfrak{A}$ is periodic, then $\mathfrak{A}$ is a direct factor of $\mathfrak{G}$. In particular, every torsion-free complete ZA-group is a direct factor of each of its periodic ZA-extensions. For a group $\mathfrak{G}$ and a natural number n, denote by $\mathfrak{G}_n$ the subgroup of $\mathfrak{G}$ generated by the nth powers of elements of $\mathfrak{G}$. The group $\mathfrak{G}$ is said to satisfy the weak condition of completeness provided that, for an arbitrary sequence of natural numbers $n_1, n_2, \cdots, n_k, \cdots$, the chain

$$\mathfrak{G} \supset \mathfrak{G}_{n_1} \supset \mathfrak{G}_{n_1 n_2} \supset \cdots \supset \mathfrak{G}_{n_1 n_2 \cdots n_k} \supset \cdots$$

consists of only finitely many distinct terms. A ZA-group satisfies the weak condition of completeness if and only if it is the product of two normal subgroups of which one is a complete ZA-group and the other is a periodic ZA-extension of a complete periodic Abelian group by a periodic group the orders of whose elements are bounded. The elements of a complete ZA-group $\mathfrak{A}$ permute with all elements of finite order in its periodic ZA-extension $\mathfrak{G}$ if and only if the maximal complete periodic subgroup of $\mathfrak{A}$ is contained in the center of $\mathfrak{G}$. A complete Abelian group $\mathfrak{A}$ is a direct factor of any extension $\mathfrak{G}$ of $\mathfrak{A}$ if and only if the commutant of $\mathfrak{G}$ intersects $\mathfrak{A}$ in the identity alone.   *R. A. Good.*

Whitehead, J. H. C. On group extensions with operators. Quart. J. Math., Oxford Ser. (2) 1, 219–228 (1950).

The theory of "kernels" as developed by Eilenberg and MacLane [Ann. of Math. (2) 48, 326–341 (1947); these Rev. 9, 7], including that of Abelian group extensions, is extended to operator groups. The chief tool is a "vector" cohomology theory. Let Y operate on Abelian G, and let X operate on both Y and G, subject to the condition (*): $x(yg) = (xy)(xg)$. For appropriate induced operators, a cochain $c_k(x_1, \cdots, x_k; y_{k+1}, \cdots, y_n)$ in $C^n(X, C^{n-k}(Y, G))$ admits two coboundary operators, δ_x and δ_y, which commute. For a vector cochain $c = (c_0, \cdots, c_n)$, the coboundary Δc is defined to have components $c_k' = \delta_x c_{k-1} + (-1)^k \delta_y c_k$. In terms of the subgroup $\mathfrak{C}_p{}^n$ of all vector cochains of type $c = (c_0, \cdots, c_p, 0, \cdots, 0)$, one defines the group

$$\mathfrak{H}_p{}^n = \mathfrak{Z}_p{}^n / \Delta \mathfrak{C}_{p-1}^{n-1}.$$

It is shown directly that: (I) The equivalence classes of X-operator extensions of Y by G constitute a group isomorphic to $\mathfrak{H}_1{}^3$. Next, Y-kernels with center G are considered, with operators X on Y and G, subject to a generalization

of (*). For these it is shown that: (II) Equivalence classes of kernels form a group isomorphic to $\mathfrak{H}_2{}^2$; and for an extendible kernel, the classes of extensions correspond one-to-one with $\mathfrak{H}_1{}^2$. The proof of (II) involves passing from a Y-kernel, with operators X, to a W-kernel, for W the appropriate split extension of X by Y, and applying to W a strengthened form of results of the reviewer [Duke Math. J. 15, 271–292 (1948); these Rev. 10, 10] concerning the (vector) cohomology groups of a group extension.

R. C. Lyndon (Princeton, N. J.).

✶**Kowalewski, Gerhard. Einführung in die Theorie der kontinuierlichen Gruppen.** Chelsea Publishing Co., New York, N. Y., 1950. viii+396 pp. $4.95.

Photographic reproduction of a book first published in 1931 by the Akademische Verlagsgesellschaft, Leipzig.

Ingram, R. E. Some characters of the symmetric group. Proc. Amer. Math. Soc. 1, 358–369 (1950).

Making use of a recurrence formula due to Murnaghan [The Theory of Group Representations, Johns Hopkins Press, Baltimore, Md., 1938] the author obtains explicit expressions for the characteristics of the classes $(2, 1^{n-2})$, $(3, 1^{n-3})$, $(4, 1^{n-4})$, $(5, 1^{n-5})$, $(2^2, 1^{n-4})$, $(3, 2, 1^{n-5})$, $(4, 2, 1^{n-6})$, $(3^2, 1^{n-6})$, $(2^3, 1^{n-6})$ of the symmetric group of order $n!$, in terms of certain numbers associated with the partition of the corresponding character. The first three are acknowledged as previously obtained by Frobenius.

D. E. Littlewood (Bangor).

Scherk, Peter. On the decomposition of orthogonalities into symmetries. Proc. Amer. Math. Soc. 1, 481–491 (1950).

E. Cartan a démontré que, dans un groupe orthogonal à n variables relatif à une forme quadratique sur le corps des nombres réels ou des complexes, toute transformation u peut s'écrire comme produit de n symétries au plus (une symétrie étant définie comme laissant invariants les points d'un hyperplan non isotrope); le rapporteur a étendu ce théorème aux formes quadratiques sur un corps quelconque de caractéristique $\neq 2$ [Sur les groupes classiques, Publ. Inst. Math. Univ. Strasbourg (N.S.) No. 1 (1945) = Actualités Sci. Ind., no. 1040, Hermann, Paris, 1948, p. 20; ces Rev. 9, 494]. L'auteur donne une nouvelle démonstration de ce théorème, qui lui fournit en même temps le nombre minimum k de symétries dont u est le produit: soit V le sous-espace dans lequel $u(x) = x$, et soit $n-m$ sa dimension; si V ne contient pas le sous-espace orthogonal V^*, on a $k = m$; dans le cas contraire (c'est-à-dire lorsque u est une transformation singulière au sens du rapporteur [loc. cit., p. 19]), on a $k = m+2$. *J. Dieudonné.*

Mautner, F. I. Unitary representations of locally compact groups. II. Ann. of Math. (2) 52, 528–556 (1950).

In a previous paper [same Ann. (2) 51, 1–25 (1950); these Rev. 11, 324], the author began the study of unitary representations of locally compact groups and dealt, in particular, with representations of discrete groups. This is a continuation of that paper and here the nondiscrete groups are treated. The main problem is formulated as follows: Let $g \rightarrow U(g)$ be a unitary representation of a locally compact group G by unitary operators $U(g)$ acting on a separable Hilbert space H, and let W be the algebra of bounded operators of H generated by all $U(g)$. If we decompose H into a direct integral $H = \int H_t$, in the sense of von Neumann

[ibid. 50, 401–485 (1949); these Rev. 10, 548], and decompose W into factors W_t accordingly, then what are the types of these factors W_t obtained in this way? If G is discrete and $U(g)$ the regular representation of G, the answer was given in the previous paper. To attack the problem in the nondiscrete case, the author refines here some former results and proves theorems on the existence of strongly continuous unitary component representations obtained by decomposing $U(g)$, on the existence of minimal invariant subspaces under certain conditions, and on the decomposition of unbounded operators into the components in H_t. It is also proved that, if a unitary representation $g \rightarrow U(g)$ of a Lie group G in a Hilbert space H is given, we can derive a representation of the Lie algebra of G by linear transformations in H, which have a common domain dense in H. By the help of these preliminaries and of Bargmann's results [ibid. 48, 568–640 (1947); these Rev. 9, 133], the author then proves that, if G is the 3-dimensional Lorentz group or the 2×2 real unimodular group, the factors W_t obtained from any strongly continuous unitary representation $U(g)$ of G are of type I almost everywhere. The author conjectures that a similar theorem might be true for any semi-simple Lie group, but the proof of this still remains an open question. Finally, it is also shown that, if $U(g)$ is the regular representation of a unimodular locally compact group, W_t is not of type III except possibly on a set of measure zero, and a formula is obtained, which is a generalization of the Peter-Weyl completeness relation and the Fourier-inversion formula in such a case. *K. Iwasawa* (Princeton, N. J.).

Segal, I. E. The two-sided regular representation of a unimodular locally compact group. Ann. of Math. (2) 51, 293–298 (1950).

Let G be a unimodular locally compact group, and let μ be its unique left and right invariant Haar measure. Let $\mathfrak{L}$ be the weakly closed algebra of bounded operators on $L_2(\mu)$ generated by left translations and let $\mathfrak{R}$ be the weakly closed algebra generated by right translations. The author proves that $\mathfrak{L}$ and $\mathfrak{R}$ are commutators of each other; that is, $\mathfrak{L}$ is the algebra of all bounded operators on $L_2(\mu)$ which commute with every operator in $\mathfrak{R}$, and conversely. It follows at once that the closed subspaces of $L_2(\mu)$ which are both right and left invariant form a Boolean algebra. This is the first step toward the proof of a generalized Plancherel theorem on arbitrary unimodular locally compact groups. *L. H. Loomis* (Cambridge, Mass.).

Segal, I. E. An extension of Plancherel's formula to separable unimodular groups. Ann. of Math. (2) 52, 272–292 (1950).

Soient G un groupe localement compact unimodulaire et séparable, dx la mesure de Haar de G, L^2 l'espace des fonctions de carré sommable pour dx. On peut définir deux représentations unitaires de G dans L^2, savoir $L_s f(x) = f(s^{-1}x)$ et $R_s f(x) = f(xs)$. Ces familles d'opérateurs engendrent respectivement des anneaux (au sens de von Neumann) $\mathfrak{L}$ et $\mathfrak{R}$; en utilisant la notation classique de von Neumann, on a $\mathfrak{L} = \mathfrak{R}'$, $\mathfrak{R} = \mathfrak{L}'$ [cf. l'analyse ci-dessus; Godement, C. R. Acad. Sci. Paris 229, 967–969 (1949); ces Rev. 11, 325; le résultat annoncé par le rapporteur est du reste plus général que celui de l'auteur]; il suit de là que l'anneau $\mathfrak{z}$ formé des opérateurs invariants à droite et à gauche est commutatif: on a en effet $\mathfrak{z} = \mathfrak{L} \cap \mathfrak{R}$. Outre les L_s, l'anneau $\mathfrak{L}$ contient aussi, pour toute $f \in L^1$, l'opérateur $L_f: g \rightarrow f * g$; on peut du reste encore définir L_f pour des $f \in L^2$ non nécessairement sommables.

Ces préliminaires posés, l'auteur montre qu'on peut définir, de façon canonique, une "weight function" [cf. von Neumann, mêmes Ann. (2) 50, 401–485 (1949); ces Rev. 10, 548] sur l'ensemble des projecteurs contenus dans $\mathcal{L}$: si P est un tel projecteur et s'il existe $f\mathcal{L}^2$ telle que $Pg = f*g$, on pose $\Delta(P) = \int |f(x)|^2 dx$; sinon on pose $\Delta(P) = +\infty$. C'est la décomposition de cette weight function en somme continue de traces "élémentaires" qui conduit au théorème de Plancherel. Étant donné que ce problème a été résolu en toute généralité par von Neumann [loc. cit.], il ne se présente pas ici de réelle difficulté, et l'auteur procède comme suit. Premièrement, on décompose L^2 en somme "mesurable" (direct integral de von Neumann) d'espaces $\mathfrak{H}(t)$ (t étant un paramètre réel) relativement à une mesure positive σ sur la droite, et ceci de telle sorte que $\mathfrak{z}$ soit exactement l'anneau des opérateurs à composantes scalaires; les anneaux $\mathcal{L}$ et $\mathcal{R}$ se décomposent alors (von Neumann) en facteurs $\mathcal{L}(t)$, $\mathcal{R}(t)$ avec $\mathcal{L}(t) = \mathcal{R}(t)'$ pour presque chaque t; pour $f\mathcal{L}^1 \cap L^2$ on note $F(t)$ la t-composante de l'opérateur $L_f \mathcal{L}$ (définie presque partout seulement; autrement dit, dans la théorie de l'auteur, la transformée de Fourier d'une fonction sommable n'est pas continue). Cela fait, l'auteur démontre tout d'abord, en observant que $\mathcal{L}$ contient "suffisamment" de projecteurs avec $\Delta(P) < +\infty$, que les $\mathcal{L}(t)$ purement infinis forment un ensemble de mesure nulle [de l'avis du rapporteur, c'est le résultat principal de cet article], la démonstration utilisant malheureusement toute la technique de von Neumann. Puis, en appliquant purement et simplement les résultats de von Neumann, on voit qu'on peut choisir pour chaque t une trace relative Tr_t dans le facteur $\mathcal{L}(t)$, de telle sorte que, pour tout projecteur P de $\mathcal{L}$, on ait $\Delta(P) = \int \mathrm{Tr}_t(P(t)) d\sigma(t)$. Il résulte de là par linéarité que, si $f, g\mathcal{L}^2$ sont combinaisons linéaires d'éléments $\varphi\mathcal{L}^2$ tels que L_φ soit un projecteur, on a $(f, g) = \int \mathrm{Tr}_t(F(t)G(t)^*) dt$, et il reste à étendre cette formule au cas de deux éléments arbitraires de L^2, ce qui ne présente pas de difficulté majeure. Si l'on considère dans $\mathcal{L}(t)$ les opérateurs A tels que $\mathrm{Tr}_t(AA^*) < +\infty$, l'expression $\mathrm{Tr}_t(AB^*)$ permet en effet, par complétion, de définir un nouvel espace de Hilbert, et la formule précédente exprime que L^2 est identifiable (canoniquement) à la somme directe de ces espaces suivant la mesure σ. On a ainsi le théorème de Plancherel sous sa forme invariante.

Evidemment, il y a lieu de croire qu'il existe une forme plus "naturelle" du dit théorème; c'est ce que le rapporteur a montré en particulier pour les groupes discrets [C. R. Acad. Sci. Paris 229, 1050–1051 (1949); ces Rev. 11, 325], de même que Gelfand et Neumark pour les groupes de Lie semi-simples. En modifiant convenablement la théorie de von Neumann, on peut du reste améliorer considérablement le résultat, de telle sorte que la "transformée de Fourier" d'une fonction sommable soit continue (et pas seulement mesurable) et, d'autre part, associée à des représentations irréductibles du groupe (l'auteur lui-même fait du reste allusion à cette possibilité). Mais le problème fondamental et non résolu consiste en ceci: Est-il possible d'associer aux composantes irréductibles de la "double représentation régulière" de G des caractères de G? Il n'est pas certain bien entendu que ce problème admette une réponse à la fois positive et raisonnable; mais il serait intéressant de l'étudier, ne serait-ce que pour éprouver la puissance des méthodes de von Neumann (convenablement modifiées au besoin); il y a en effet quelque chance pour que ces méthodes jouent par la suite un rôle capital en analyse fonctionnelle et s'appliquent à toute sorte d'autres problèmes que ceux de la théorie des groupes.   *R. Godement* (Nancy).

Yamabe, Hidehiko.   On an arcwise connected subgroup of a Lie group.   Osaka Math. J. **2**, 13–14 (1950).

This paper gives a direct proof of the theorem that an arcwise connected subgroup of a Lie group is a Lie subgroup. According to the author, an independent proof has been found by Kuranishi based on a similar result for the vector group and for semisimple groups.

D. Montgomery (Princeton, N. J.).

Hall, Marshall, Jr.   A topology for free groups and related groups.   Ann. of Math. (2) **52**, 127–139 (1950).

Étant donné un élément g d'un groupe libre G, il existe un sous-groupe d'indice fini dans G qui ne contient pas g; cette propriété permet d'introduire dans tout groupe G qui la possède une topologie non triviale, obtenue en prenant comme système fondamental de voisinages de l'unité les sous-groupes d'indice fini de G. Dans cette topologie, un sous-groupe est fermé seulement s'il est intersection de sous-groupes d'indice fini, ce qui n'est pas toujours le cas, comme l'auteur le montre par un exemple. La topologie de G peut du reste aussi bien être définie par la famille (F_α) des sous-groupes invariants d'indice fini; le groupe $\hat{G}$ obtenu en complétant G s'identifie à la limite projective des groupes (finis) G/F_α, en sorte que $\hat{G}$ est compact et totalement discontinu. Dans le cas où G est à un nombre fini de générateurs, le groupe $\hat{G}$ est métrisable, donc identifiable, en tant qu'espace topologique, à l'ensemble de Cantor. L'auteur démontre que cette identification peut être faite de telle sorte qu'elle transforme la mesure de Haar sur $\hat{G}$ en la mesure de Lebesgue sur la droite. L'auteur donne enfin quelques applications de ces méthodes, en particulier une démonstration simple d'un résultat de W. Magnus [J. Reine Angew. Math. **177**, 105–115 (1937)]: Les commutateurs successifs d'un groupe libre ont pour intersection l'élément unité du groupe.

R. Godement (Nancy).

Graev, M. I.   On free products of topological groups.   Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 343–354 (1950).   (Russian)

The author defines the free product (as opposed to the direct product) of an arbitrary collection M of topological groups A_α ($\alpha\epsilon M$) by these properties: (1) G is a topological group with A_α as subgroup, $\alpha\epsilon M$; (2) the minimal closed subgroup of G containing every A_α coincides with G; (3) if H is a topological group and there are given continuous homomorphisms h_α of A_α into H ($\alpha\epsilon M$), then there exists a continuous homomorphism h of G into H coinciding on each A_α with the given h_α. The author proves that a group G with these properties exists, whatever the collection of groups A_α, which is unique to within an isomorphism keeping elements of each A_α fixed. If M_1 is a subset of M, and A is the group generated by the corresponding A_α, $\alpha\epsilon M_1$, and B is the group generated by the remaining A_α, $\alpha\epsilon M - M_1$, then A and B are closed and G is their free product. Further, if G is a free product of groups A_α, each of these a free product of groups $A_{\alpha\beta}$, then G is a free product of the totality $A_{\alpha\beta}$. The author's proof of the existence of G is such as to show also that the group G is free in the algebraic sense, if topologies are ignored in the A_α and in G. Almost all of the argument of the paper is devoted to the case of two factors; the rest is relatively easy.

L. Zippin.

Melencov, A. A. Cuts in connected topological groups. Doklady Akad. Nauk SSSR (N.S.) **72**, 845–847 (1950). (Russian)

Let G be a connected topological group and H a subgroup with identity component N. If $G-H$ is not connected, H is a cut. If, furthermore, H/N is discrete and $G-H+N$ is connected, then the cut H is a simple cut provided that a certain symmetric connected neighborhood Q of N can be found in which N is an irreducible separation. The set-up is somewhat unusual when H is not connected, even when G is locally connected. A connected cut is always simple. For a simple cut H, in the case that G is locally connected, $G-H$ is uniquely the sum of two connected open sets P and P^{-1}, each the inverse of the other. In this case, if Q is a neighborhood of N, every finite cyclic subgroup of G which belongs to Q is contained in N; if H is a connected cut, then every element of finite order belongs to H. The set P, above, is invariant under transformations defined by $y=nxn'$ with n and n' in N. If H is an invariant simple cut (G locally connected and connected), the set P is invariant too, and has properties analogous to those of the positive reals. Here the author introduces a concept of "weakly Archimedean with respect to a normal subgroup" and states a necessary and sufficient condition that G/H be isomorphic to the additive group of reals. The proofs are said to be based upon a lemma of G. T. Whyburn [Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942, chapter 3, p. 42; these Rev. **4**, 86].   *L. Zippin.*

NUMBER THEORY

✠**Gloden, A. Table des bicarrés N^4 pour $3000 < N \leq 5000$.** 2d ed. Published by the author, Luxembourg, 1950. 19 pp. 50 Belgian francs.

New edition without change of the tables published by the author in 1947; see these Rev. **9**, 207.

Gloden, A. Note d'analyse diophantienne. Sur l'équation biquadratique $x_1^4+x_2^4+x_3^4=y_1^4+y_2^4+y_3^4$. Bull. Inst. Polytech. Jassy [Bul. Inst. Politech. Iaşi] **4**, 54–57 (1948).

Various two parameter families of solutions of the equation indicated in the title are given.   *I. Niven.*

Xeroudakes, Georgios F. On some problems of indeterminate analysis. Bull. Soc. Math. Grèce **24**, 28–50 (1949). (Greek)

Summaries of known results on various problems in Diophantine analysis. The early part of the paper deals with some examples studied by Fermat which arise from seeking Pythagorean right triangles whose sides satisfy certain auxiliary conditions. Among the other problems discussed are the equations $x^4+y^4+\alpha^2z^4=X^4+\psi^4+\alpha^2Z^4$, and $x^4+mx^2y^2+ny^4=kz^2$, with complete solutions given in some special cases.   *T. M. Apostol* (Pasadena, Calif.).

Beeger, N. G. W. H. On composite numbers n for which $a^{n-1}=1 \pmod{n}$ for every a prime to n. Scripta Math. **16**, 133–135 (1950).

Carmichael [Amer. Math. Monthly **19**, 22–27 (1912)] has proved that a composite number n satisfies the condition stated in the title of this paper if and only if n is the product of 3 or more different odd primes p_i and each p_i-1 divides $n-1$. In the present paper it is proved that, in case n is a product $p_1p_2p_3$, only a finite number of values of p_2 and p_3 can be employed with fixed p_1.   *I. Niven.*

Adelman, Donald M. Note on the arithmetic of bilinear transformations. Proc. Amer. Math. Soc. **1**, 443–448 (1950).

Let the sequence $\{x_i\}$ be determined by x_1 and the relation $x_{n+1}=(ax_n+b)/(cx_n+d)$, $ad-bc\neq0$, and a, b, c, d integers. It is tacitly assumed that $c\neq0$. It is shown that if the x_i are all integers then $\{x_i\}$ is periodic with period 1, 2, 3, 4, or 6. A method is found for obtaining all x_1, a, b, c, d which yield integral sequences $\{x_i\}$.   *H. S. Zuckerman.*

Yakovkin, M. V. On some criteria for irreducibility of polynomials. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 283–295 (1950). (Russian)

The author proves three theorems which give criteria for irreducibility of polynomials and states numerous corollaries.

The first theorem states that, if $f(x)$ is a polynomial of degree n with integral coefficients which possesses a divisor $\varphi(x)$ of degree m ($1 \leq m < n$) whose coefficients are rational, then the integer $q^n f(p/q)$ must decompose into two factors of magnitudes exceeding $|qt|^m$ and $|qt|^{n-m}$, where p/q is any rational number distant more than t from the real part of each root of $f(x)$. This is a generalisation of a previous result of the author [Doklady Akad. Nauk SSSR (N.S.) **58**, 1915–1918 (1947); these Rev. **9**, 331]. Theorem 2 is the same as theorem 2 of an earlier paper [ibid. (N.S.) **66**, 169–172 (1949); these Rev. **11**, 11], while theorem 3 is a slightly more general form of a criterion also given earlier by the author [ibid. (N.S.) **64**, 771–774 (1949); these Rev. **10**, 591].   *R. A. Rankin* (Cambridge, England).

Ward, Morgan. Arithmetical properties of polynomials associated with the lemniscate elliptic functions. Proc. Nat. Acad. Sci. U. S. A. **36**, 359–362 (1950).

The author [Amer. J. Math. **70**, 31–74 (1948); these Rev. **9**, 332] has studied the arithmetical properties of polynomials associated with the real multiplication of elliptic functions. This communication is a report of the extension of this study to complex multiplication in the special case where the Weierstrass invariant g_2 is zero. Further results and proofs are to be published elsewhere.

    H. S. Zuckerman (Seattle, Wash.).

Deuring, Max. Die Anzahl der Typen von Maximalordnungen einer definiten Quaternionenalgebra mit primer Grundzahl. Jber. Deutsch. Math. Verein. **54**, 24–41 (1950).

In a former paper on the structure of rings of multiplicators of (abstract) elliptic function fields [Abh. Math. Sem. Hansischen Univ. **14**, 197–272 (1941); these Rev. **3**, 104], the author derived the following formulas:

(i) $$\sum h(d_{p'})+h_p=p' \qquad (f=2,4,6,\cdots)$$
$$\sum h(d_{p'})+2t_p-h_p=p' \qquad (f=1,2,3,\cdots),$$

where p is a prime, $-d_{p'}$ runs over all discriminants prime to p of positive binary quadratic forms whose principal forms represent p' properly, $h(d)$ denotes the class number of primitive binary quadratic forms with discriminant $-d$, and t_p, h_p denote, respectively, the class number and the number of types of maximal orders of the quaternion algebra with $-p$ as ground-number. Combining this with the classical class number relation of 1st "Stufe" [F. Klein and R. Fricke, Vorlesungen über die Theorie der elliptischen Modulfunktionen, v. II, Teubner, Leipzig, 1892, p. 184], the author

gives the following formulas for h_p and t_p:

(ii) $h_p = p/12 + \tfrac{1}{2} - \tfrac{1}{3}(-3/p) - \tfrac{1}{4}(-4/p)$,

(iii) $t_p = \tfrac{1}{3}p + \tfrac{1}{2} + \tfrac{1}{3}(-3/p) + \tfrac{1}{4}(-4/p)$

$\qquad + \begin{cases} \tfrac{1}{2}h(\sqrt{-p}) & \text{for } p \equiv 1 \pmod 4, \\ \tfrac{1}{2}h(\sqrt{-p})(3 - (-\tfrac{1}{2}p)) & \text{for } p \equiv -1 \pmod 4,\ p > 3. \end{cases}$

Formula (ii) has been found by a different method by M. Eichler [Math. Z. **43**, 102–109 (1937)]. In the present proof the author expresses first the class number sum in the class number relation of 1st "Stufe" by means of primitive class numbers, and then combines the so obtained formula with the case $f = 2$ of (i), which gives, after detailed study of the number of representations of an integer by a principal form, the formula (ii). The case $f = 1$ leads to (iii). The method is compared with the measure-theoretical one. The rôle of (iii) in the theory of modular functions is discussed by constructing h_p theta-series belonging to h_p ideal classes in our quaternion algebra. It is shown further that (i) with $f \geqq 3$ gives no new result.   *T. Nakayama* (Urbana, Ill.).

Nielsen, Jakob. A study concerning the congruence subgroups of the modular group. Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 18, 32 pp. (1950).

The author presents a method of determining a system of generators and relations for the principal congruence subgroup modulo q of the full modular group. Here q is a prime > 5 subject to the restriction that $r = \tfrac{1}{2}(q-1)$ is the least positive exponent for which $2^r \equiv \pm 1 \pmod q$. His method is based on the fact that the fundamental region of a subgroup of the modular group (of finite index) is a polygon bounded by straight lines and circular arcs which are pairwise congruent under certain substitutions of the subgroup; these substitutions constitute a system of generators.

The author starts with a set of $r(q+1)$ q-sided polygons, which he combines into a two-dimensional closed orientable manifold Φ by introducing certain identifications of the sides based on the arithmetic of the moduli q and r. He then finds a set of generators and relations for the fundamental group G of Φ. Now let F be the abstract group on the letters S, T with the relations (1) $S^q = 1$, $T^3 = 1$, $(ST)^3 = 1$, and H the normal subgroup generated by certain elements $\{k\}$ of F. Then H is shown to be isomorphic to G, and a system of generators of H is obtained from those of G. The isomorphism is demonstrated by realizing H as a group of non-Euclidean motions in the plane which generate the universal covering surface of Φ. The final step is to identify S with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, T with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and consider the elements of the matrices modulo q. The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \pmod q$ is by definition C, the principal congruence subgroup modulo q. The author now translates the abstract groups F, H, F/H into groups of matrices through the above identifications and, by using elementary group-theoretic devices, arrives at the following result. The set of matrices $\{mS^vm^{-1}\}$ and $\{k\}$ constitute a system of generators for C, where m runs over a finite set of mutually incongruent matrices modulo q, and $\{k\}$ are the generators of H expressed as matrices. The relations among the generators are obtained from (1) and the relations among the $\{k\}$ in H, and this system of relations is capable of further reduction which, however, the author does not attempt. The author points out that most of his results were obtained previously by Frasch [Math. Ann. **108**, 229–252 (1933)], who used the group-theoretic method of Reidemeister and Schreier.   *J. Lehner* (Philadelphia, Pa.).

Walton, L. F. Ideal numbers over integral domains having non-maximal prime ideals. Duke Math. J. **17**, 285–298 (1950).

Prüfer's theory of ideal numbers for an algebraic number field K of finite degree [Math. Ann. **94**, 198–243 (1925)] was developed in a different way and extended in some directions by von Neumann [Acta Litt. Sci. Szeged **2**, 193–227 (1926)], who assumed the Dedekind theory of ideals for the integral domain D of all integral elements of K, and defined an ideal number $\mathfrak{A}$ as a class of equivalent fundamental sequences of elements of K. He called a sequence $R = [\alpha_1, \alpha_2, \cdots]$ fundamental if $\alpha_r - \alpha_{r+1}$ has the factor $\mathfrak{p}^m$ for all prime ideals $\mathfrak{p}$ of D and all positive integers m, for almost all $r = 1, 2, \cdots$, and called R and $S = [\beta_1, \beta_2, \cdots]$ equivalent if almost all $(\alpha_r - \beta_r)$ have the factor $\mathfrak{p}^m$ for all $\mathfrak{p}$ and m. An element α of K is said to have the factor $\mathfrak{p}^m$ if there exists an element $\xi \epsilon D$ such that ξ is not divisible by $\mathfrak{p}$ and $\alpha\xi$ is divisible by $\mathfrak{p}$ in the usual sense. Addition and multiplication may be defined in the usual way and then the set of all ideal numbers over D forms a ring which contains a proper subring isomorphic to K, and in which there is the expected type of limit-closure. An ideal number $\mathfrak{A}$ is said to have the factor $\mathfrak{p}^m$ if for any sequence $[\alpha_1, \alpha_2, \cdots] \epsilon \mathfrak{A}$ almost all α_r have the factor $\mathfrak{p}^m$. An ideal number $\mathfrak{B}$ is called a $\mathfrak{p}$-adic ideal number if $\mathfrak{B}$ has the factors $\mathfrak{q}^m$, for all m and all prime ideals $\mathfrak{q} \neq \mathfrak{p}$. Von Neumann proved that every ideal number $\mathfrak{A}$ has uniquely determined so-called $\mathfrak{p}$-adic components $(\mathfrak{A})_{\mathfrak{p}}$, for all $\mathfrak{p} \epsilon D$, and that

$$\mathfrak{A} = \sum_{r=1}^{\infty} (\mathfrak{A})_{\mathfrak{p}} = \prod_{r=1}^{\infty} (1 + (\mathfrak{A} - 1)_{\mathfrak{p}}).$$

His three Hauptsätze have to do with product representations of ideal numbers, exponent-sequences defined by these, and relations between divisibility in D and in the ring of ideal numbers over D.

It is well known that necessary and sufficient conditions for the Dedekind theory in an integral domain D with quotient-field K are: (1) the Teilerkettensatz holds for ideals of D; (2) every nonzero prime ideal of D is divisorless (i.e., maximal in D); (3) D is integrally closed in K [cf. van der Waerden, Moderne Algebra, vol. 2, 2d ed., Springer, Berlin, 1940, pp. 84–93; these Rev. **2**, 120]. If for D and K, (1) and (3) only are postulated, van der Waerden and Artin have shown [ibid., pp. 93–97] that the analogue of the Dedekind unique-factorization-theorem holds in terms of quasi-equality for ideals: $\mathfrak{a} =' \mathfrak{b}$ [or $\mathfrak{a} \sim \mathfrak{b}$], meaning that $\mathfrak{a}^{-1} = \mathfrak{b}^{-1}$, and with restriction to higher prime ideals: $(\mathfrak{p}^{-1})^{-1} = \mathfrak{p}$. From their results follow analogues of other well-known theorems of the Dedekind theory; for example, if $\mathfrak{a}$ and $\mathfrak{b}$ are D-ideals of K, $\mathfrak{b} \subset D$, then there exists an ideal $\mathfrak{c}$ in D such that $\mathfrak{a}\mathfrak{c} =' (q)$, q in K, and $(\mathfrak{b}, \mathfrak{c}) =' D$. The paper under review extends von Neumann's theory of ideal numbers to D's satisfying (1) and (3) only, employing the van der Waerden-Artin theory instead of Dedekind's. The ideal-theory is first summarized, with proofs here and there, and then von Neumann's theory is developed, mutatis mutandis, as far as possible. At an appropriate stage the additional restriction on D is made that its set of higher prime ideals is denumerable, and the relation $\mathfrak{A} = \sum_{r=1}^{\infty} (\mathfrak{A})_{\mathfrak{p}}$ is obtained. The product representations, and hence the three Hauptsätze, do not go over into the more general case.

On p. 292 the author gives a correct proof of a theorem for which von Neumann's argument is false [loc. cit., p. 205]. On p. 296 the author alleges that von Neumann asserts that: $\mathfrak{p}^e | \alpha/\beta$, α, $\beta \epsilon D$, implies $\mathfrak{p} \nmid \beta$, and gives the counter-example

in the quadratic algebraic number field containing $i\sqrt{5}$: $(3, 2+i\sqrt{5})^0 | \frac{1}{3}(2+i\sqrt{5}), ((3), (3, 2+i\sqrt{5})) \neq D|$. The reviewer cannot find the alleged assertion in the place cited, but only the assertion [loc. cit., pp. 196, 210] that if $\mathfrak{p}^0$ is a factor of $\alpha\mathfrak{k}K$, then $\alpha = \kappa/\lambda$, for some κ, λ in D and λ not divisible by $\mathfrak{p}$. Since $\frac{1}{3}(2+i\sqrt{5}) = 3/(2-i\sqrt{5})$ and $(3, 2+i\sqrt{5}, 2-i\sqrt{5}) = D$, the reviewer's verdict in this instance is for von Neumann. The author himself omitted the hypothesis $\mathfrak{b} \subset D$ in the example cited above.

R. Hull (Lafayette, Ind.).

Vinogradov, I. M. The upper bound of the modulus of a trigonometric sum. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 199–214 (1950). (Russian)

The author gives a detailed, self-contained proof of the following theorem. Suppose $f(x) = a_{n+1}x^{n+1} + \cdots + a_1 x$ is a polynomial with real coefficients of degree $n+1 \geq 12$, m and p are positive integers, and $S = \sum_{x=1}^{p} \exp\{2\pi i m f(x)\}$. Let r be one of the numbers 2, 3, $\cdots$, $n+1$ and suppose that $a_r = a/q + \theta/q^2$, where a and q are integers, $1 < q < p^r$, $(a, q) = 1$ and $|\theta| \leq 1$. Put $\tau = (\log q)/(\log p)$ if $1 < q < p$, $\tau = 1$ if $p \leq q \leq p^{r-1}$, and $\tau = r - (\log q)/(\log p)$ if $p^{r-1} < q < p^r$. Then $|S| < 8n^{\frac{1}{2}n^2l}m^{2\rho/\tau}p^{1-\rho}$, where $l = \ln\{12n(n+1)/\tau\}$ and $\rho = \tau/(3n^2l)$. Previous papers of the author and others have considered only the case $r = n+1$. *P. T. Bateman.*

Vinogradov, I. M. A new improvement of the method of estimation of double sums. Doklady Akad. Nauk SSSR (N.S.) 73, 635–638 (1950). (Russian)

This paper is concerned with two separate questions, and the only common feature is that in both cases the method involves "exhausting" the region of summation of a double sum. The first result is as follows. Let χ be a nonprincipal character (mod q), and let k be a fixed integer, positive or negative (but not zero). Then

(1) $$\sum_{p \leq N} \chi(p+k) = O(Nq^{\epsilon}(N^{-1}q^{5/6})^{1/4}),$$

provided that $q^{5/6} < N < q^{7/6}$, the summation being over primes p. An earlier result [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 7, 17–34 (1943); these Rev. 5, 143] gave an estimate which was effective for $q > N^{1+\delta}$ ($\delta > 0$). The proof is given only in outline, and the reviewer has not been able to fill in the details in the proofs of lemmas 2 and 3 or of the main result (1). The second result relates to the straightforward character sum $\sum_{N+1}^{N+Q} \chi(x)$. Let (2) $U(M, P) = \sum_{x=M+1}^{M+P} \sum_{s=0}^{P-1} \chi(x+s)$. Let $M_1, \cdots, M_m$ be integers such that the intervals $M_i \leq t \leq M_i + P$ are nonoverlapping and are contained in $M_1 \leq t \leq M_1 + q$. Then it is easily proved, using Cauchy's inequality, that (3) $\sum_{i=1}^{M} |U(M_i, P)| < P(mq)^{\frac{1}{2}}$. The sum $U(M, P)$ can be regarded as the sum of $\chi(x)$ over all points (x, y) of the parallelogram whose vertices are $(M, 0)$, $(M+P, 0)$, $(M+P, P)$, $(M+2P, P)$; and (3) estimates the sum over a series of such parallelograms, subject to the condition that their lower sides do not overlap and have a total length not more than q. Now put $Q = hq^{\frac{1}{2}}$, $\tau = [\log h/\log 3]$, $Q_0 = 2 \cdot 3^{\tau}[3^{-\tau}Q/2]$, and suppose for simplicity that $\tau \leq 1$. Let $S_0 = \sum_{N+1}^{N+Q_0} \chi(x)$. Then $Q_0 S_0$ is $\sum \chi(x)$ summed over the square $N < x \leq N + Q_0$, $0 < y \leq Q_0$. This square is approximated to by parallelograms. The first parallelogram is $N - \frac{1}{2}Q_0 < x - y \leq N + \frac{1}{2}Q_0$, $0 < y \leq Q_0$. Next there are four parallelograms near the corners of the square, similar to $N - \frac{1}{4}Q_0 < x - y \leq N - \frac{1}{4}Q_0$, $0 < y \leq \frac{1}{2}Q_0$. Generally there are $4 \cdot 3^{s-1}$ parallelograms of side $3^{-s}Q_0$. The sums of $\chi(x)$ over these are to be taken with appropriate $+$ and $-$ signs.

The sum over a single parallelogram is equivalent to one of the type (2), and the sum over a group of parallelograms can be estimated by (3). The final result is that (4) $|\sum_{N+1}^{N+Q} \chi(x)| < q^{\frac{1}{2}}(a \log h + b)$, where $a = 2/\log 27$ and $b = 5/2$. But this result is less interesting than the ingenious and elementary method of proof. *H. Davenport.*

Rényi, Alfréd. Probability methods in number theory. Publ. Math. Collectae Budapest 1, no. 21, 9 pp. (1949).

This paper is the author's inaugural lecture at the University of Budapest. First a brief summary of some applications of probabilistic methods to number theory are given. These also include incorrect applications which the author analyzes and criticizes. The major part of the paper is devoted to the exposition of the author's approach to the "large sieve." The details of this approach have been published elsewhere [J. Math. Pures Appl. (9) 28, 137–149 (1949); Compositio Math. 8, 68–75 (1950); these Rev. 11, 161, 581]. *M. Kac* (Ithaca, N. Y.).

Hlawka, Edmund. Über Gitterpunkte in Parallelepipeden. J. Reine Angew. Math. 187, 246–252 (1950).

The author's main result is as follows. Suppose we are given n independent directions in n-dimensional space, and a positive number V. Then there exists a parallelepiped P, with centre O and volume V, and with its faces normal to the given directions, such that the number of pairs of opposite points with integral coordinates lying in P (not counting O) is less than $A_n V$. Here A_n is the constant $(1/n)(n!)^2 2^{\frac{1}{2}n(n-1)}$. The proof is by an extension of the method by which Siegel [see Davenport, Acta Arith. 2, 262–265 (1937)] proved that such a parallelepiped exists, with volume depending only on n, which contains no point with integral coordinates except O. A new feature is the definition (§ 3) of successive minima depending on a number l. The author also proves a theorem relating to any convex body with centre O. This could in fact be deduced from the earlier result, and with a better constant, by circumscribing a parallelepiped to the convex body [see Dvoretzky and Rogers, Proc. Nat. Acad. Sci. U. S. A. 36, 192–197 (1950), theorem 5A; these Rev. 11, 525]. *H. Davenport.*

Davenport, H., and Rogers, C. A. On the critical determinants of cylinders. Quart. J. Math., Oxford Ser. (2) 1, 215–218 (1950).

Let K be a plane star body, symmetric with respect to the origin, and C the cylindrical star body of points (x, y, s), where $|s| \leq 1$ and (x, y) is a point of K. Let $\Delta(K)$, $\Delta(C)$ denote the critical determinants of K and C respectively. It is clear that $\Delta(C) \leq \Delta(K)$. Recently Rogers [same J., Oxford Ser. (1) 20, 45–47 (1949); these Rev. 11, 83] constructed a star body K with $\Delta(C) < \Delta(K)$. In this note another star body K is constructed so that $\Delta(C)/\Delta(K)$ is arbitrarily small. *D. Derry* (Vancouver, B. C.).

Chalk, J. H. H. On the frustrum of a sphere. Ann. of Math. (2) 52, 199–216 (1950).

Let K_λ be the truncated sphere

$$\max\ ((x^2+y^2+z^2)^{\frac{1}{2}}, |z|/\lambda) \leq 1,$$

where $0 < \lambda \leq 1$. Then the critical determinant of K_λ is (1) $\Delta(K_\lambda) = \frac{1}{2}\lambda(3-\lambda^2)^{\frac{1}{2}}$. The only critical lattices (i.e., lattices with determinants equal to $\Delta(K_\lambda)$ having no point other than $(0, 0, 0)$ in the interior of K_λ) are the lattices which may be obtained, by means of rotations about the z-axis

and reflexions in the coordinate planes, from the lattice

$$x = v - \tfrac{1}{2}w, \quad y = \tfrac{1}{2}w(3 - \lambda^2)^{\frac{1}{2}}, \quad z = \lambda u + \tfrac{1}{2}\lambda w$$

$(u, v, w = 0, \pm 1, \pm 2, \pm 3, \cdots)$. As a simple corollary we get the following result of K. Mahler [Quart. J. Math., Oxford Ser. (1) **17**, 16–18 (1946); these Rev. **7**, 368]: The critical determinant of the cylinder max $((x^2 + y^2)^{\frac{1}{2}}, |z|) \leq 1$ is equal to $3^{\frac{1}{2}}/2$. Another corollary of (1): Let $F(x, y, z)$ be a positive definite quadratic form of determinant D. Let α, β, γ be real numbers satisfying $\bar{F}(\alpha, \beta, \gamma) > D$, where $\bar{F}$ is the adjoint form; put $\bar{F}(\alpha, \beta, \gamma) = \lambda^{-2}D$, $\lambda > 0$. Then there are integers x, y, z, not all zero, such that

$$\max (F(x, y, z), |\alpha x + \beta y + \gamma z|^2) \leq 2^{\frac{1}{3}}D^{\frac{1}{3}}\lambda^{-\frac{2}{3}}(3 - \lambda^2)^{-\frac{1}{3}}.$$

V. Jarník (Prague).

Descombes, Roger, et Poitou, Georges. Sur l'approximation dans $R(i\sqrt{11})$. C. R. Acad. Sci. Paris **231**, 264–266 (1950).

The authors call $C = \inf_x \lim \sup_{p, q} (|q(p - qx)|)^{-1}$, where p and q range over the integral elements of a quadratic field $R(i\sqrt{m})$, and where x ranges over all complex numbers, the Hurwitz constant of $R(i\sqrt{m})$. For $m = 1, 2, 3,$ and 7, $C = 3^{\frac{1}{2}}, 2^{\frac{1}{2}}, 13^{\frac{1}{2}},$ and $8^{\frac{1}{2}}$, respectively. [For references, see Koksma, Diophantische Approximationen, Springer, Berlin, 1936, pp. 51–52; Hofreiter, Monatsh. Math. Phys. **45**, 175–190 (1937).] For $m = 11$, it has been known that $C \leq 5^{\frac{1}{2}} = 1.495 \cdots$. The authors find that $C = 5^{\frac{1}{2}}/2 = 1.118 \cdots$, but sketch their methods of proof very briefly. An extension of the theory of continued fractions is required, different from that of Hurwitz which does not yield "sequences of best approximation" for certain complex numbers x_0. The proofs also require the Euclidean algorithm which exists in the five cases $m = 1, 2, 3, 7,$ and 11.   *R. Hull.*

Cassels, J. W. S. Some metrical theorems in Diophantine approximation. I. Proc. Cambridge Philos. Soc. **46**, 209–218 (1950).

Let $f_{1,j}(\theta_j), \cdots, f_{n,j}(\theta_j), j = 1, 2, \cdots, w$, be w sequences of differentiable functions, defined for $a_j \leq \theta_j \leq b_j$. Assume that there is an absolute constant K so that for $m \neq n$ $|f'_{m,j}(\theta_j) - f'_{n,j}(\theta_j)| \geq K > 0$, also that $f'_{m,j}(\theta_j) - f'_{n,j}(\theta_j)$ is monotonic. Let $0 \leq \beta_j \leq \gamma_j \leq 1$; denote by $F_N(\beta, \gamma; \theta)$ the number of $n \leq N$ so that simultaneously $\beta_j \leq f_{n,j}\theta \leq \gamma_j$, $j = 1, 2, \cdots, w$. The author proves the following theorem. For almost all sets θ_j,

$$\left| F_N(\beta, \gamma; \theta) - N \prod_{j=1}^{w} (\gamma_j - \beta_j) \right| < N^{\frac{1}{2}}(\log N)^{w + \frac{1}{2} + \epsilon}$$

holds for all $\beta, \gamma,$ and all $N > N_0$, where N_0 depends only on θ and ϵ. An interesting corollary results when one takes $w = 1$ and $f_n(\theta) = n_n\theta$, where $n_1 < n_2 < \cdots$ is a sequence of integers. A similar theorem has been found almost simultaneously by Koksma and the reviewer [Nederl. Akad. Wetensch., Proc. **52**, 851–854 = Indagationes Math. **11**, 299–302 (1949); these Rev. **11**, 331].   *P. Erdős* (Aberdeen).

Cassels, J. W. S. Some metrical theorems of Diophantine approximation. II. J. London Math. Soc. **25**, 180–184 (1950).

[For part I cf. the preceding review.] Generalising previous results of Khintchine and Koksma the author proves the following result: Let $f_n(\theta)$, $n = 1, 2, \cdots$, be an infinite sequence of differentiable functions defined in the interval $(0, 1)$. We assume that $f_n'(\theta) > 0$, $f_n'(\theta)$ is monotone increas-

ing in θ for fixed n and $\lim_{n \to \infty} f_n'(\theta) = \infty$ for fixed θ. Further let $0 \leq \varphi_n < 1$, $\psi_n = \sum_{k=1}^{n} \varphi_k$. The author remarks that if $\sum \varphi_n$ converges then (1) $\{f_n(\theta)\} < \varphi_n$ has only a finite number of solutions for almost all θ ($\{x\}$ denotes the fractional part of x). The proof is easy. His second result is very much deeper. Write

$$F_n(\theta) = (f_n'(\theta))^{-1} \sum_{k=1}^{n} f_k'(\theta), \quad E_n = \int_0^1 F_n(\theta) d\theta.$$

Assume that $\sum \varphi_n$ diverges and that

$$\sum_{k=1}^{n} \varphi_k E_k = o(\psi_n^2), \quad \sum \varphi_n n / f_n'(0) = o(\psi_n^2).$$

Then the inequality (1) has infinitely many solutions for almost all θ. Interesting special cases are obtained by putting $f_k(\theta) = n_k\theta$.   *P. Erdős* (Aberdeen).

Cassels, J. W. S. Some metrical theorems of Diophantine approximation. III. Proc. Cambridge Philos. Soc. **46**, 219–225 (1950).

[For part II cf. the preceding review.] Let $\psi(n) > 0$ be any function of the positive integer n. The author first of all proves the following theorems. The inequality $0 \leq n\theta - m - \alpha \leq \psi(n)$ has an infinity of integer solutions $n > 0$ and m for almost all or almost no sets of numbers θ, α according as $\sum \psi(n)$ diverges or converges ("almost all" is understood in the sense of two-dimensional measure). If $\psi(n)$ is monotonically decreasing, then the inequality $0 \leq n\theta - m < \psi(n)$ has an infinity of integer solutions $n > 0$ and m for almost all or almost no θ according as $\sum \psi(n)$ diverges or converges. The condition of monotonicity cannot be omitted. Let λ_n be an increasing sequence of integers. Denote by μ_n the number of fractions of the form k/λ_n, $0 < k < \lambda_n$, which are not of the form j/λ_m, $m < n$. The author defines λ_n to be a $\sum$ sequence if $\lim \inf N^{-1} \sum_{n \leq N} \mu_n \lambda_n^{-1} > 0$. The author proves that if $\psi(n)$ is monotonically decreasing and λ_n is a $\sum$ sequence, then the inequality $0 \leq \lambda_n\theta - m < \psi(n)$ has an infinity of integer solutions $n > 0$ and m for almost all or for almost no θ according as $\sum \psi(n)$ diverges or converges. He also shows that not all sequences are $\sum$ sequences.   *P. Erdős.*

Cassels, J. W. S. Some metrical theorems of Diophantine approximation. IV. Nederl. Akad. Wetensch., Proc. **53**, 176–187 = Indagationes Math. **12**, 14–25 (1950).

The author improves an earlier result [see the third preceding review] which had also been obtained by Erdős and Koksma [same Proc. **52**, 851–854 = Indagationes Math. **11**, 299–302 (1949); these Rev. **11**, 331] and proves the following result. Let $\varphi(n)$ be a positive function of the positive integer n such that $\log n \log \log n \geq \varphi(n) \geq c > 0$ (c independent of n) for all sufficiently large n, and that $\varphi(n)$ and $\varphi(n)^{-1} \log n \log \log n$ are finally monotonic nondecreasing. Let $f_n(\theta)$ ($n = 1, 2, 3, \cdots$) have positive monotonic nondecreasing continuous derivatives for $0 \leq \theta \leq 1$. For $0 \leq \alpha \leq \beta \leq 1$ denote by $F_N(\alpha, \beta, \theta)$ the number of integers n for which $\alpha \leq f_n(\theta) < \beta$ (mod 1), and put

$$\Re_N(\theta) = \max_{\alpha, \beta} |F_N(\alpha, \beta, \theta) - N(\beta - \alpha)|.$$

Then, if $f'_n(\theta) \geq e^{\varphi(n)} f'_{n-1}(\theta)$ ($n = 2, 3, \cdots$), $f'_1(\theta) \geq 1$, for all θ, there is an absolute positive constant A_0 such that

$$\Re_N(\theta) \leq A_0 N^{\frac{1}{2}} \varphi(N)^{-\frac{1}{2}} (\log N)^{\frac{1}{2}} \log \log N$$

for almost all θ and all sufficiently large N.

K. Mahler (Manchester).

LeVeque, Wm. J. Note on a theorem of Koksma. Proc. Amer. Math. Soc. **1**, 380–383 (1950).

The theorem mentioned in the title [Compositio Math. **2**, 250–258 (1935)] states that under certain general conditions for the functions $g(x, n)$, $a \leq x \leq b$, $n = 1, 2, \cdots$, the sequence (1) $g(x, 1)$, $g(x, 2)$, $\cdots$ is uniformly distributed (mod 1) for almost all values of x, $a \leq x \leq b$. Using a lemma of Kac, Salem, and Zygmund [Trans. Amer. Math. Soc. **63**, 235–243 (1948); these Rev. **9**, 426], the author provides a new proof for a general, important case of that theorem. With his method the author proves further, that under his assumptions, for almost all x in $a \leq x \leq b$,

$$\sum_{n=1}^{N} e^{2\pi i m g(x, n)} = o(N^{1+\theta}),$$

$\theta > 0$, m an integer $\neq 0$. The reviewer remarks that this result is contained in theorems of Erdös and Koksma [Nederl. Akad. Wetensch., Proc. **52**, 264–273, 851–854 = Indagationes Math. **11**, 79–88, 299–302 (1949); these Rev. **11**, 14, 331] and Cassels [see the preceding review]; they proved among other things (with different methods), that (under general conditions) $ND(N) = o(N^{\frac{1}{2}} \log^{3/2 + \theta} N)$, $\theta > 0$, for almost all x, $a \leq x \leq b$. Here $D(N)$ means the discrepancy in the uniform distribution of the sequence (1) [cf. Erdös and Koksma, loc. cit.]. *J. F. Koksma* (Amsterdam).

Černý, Karel. Contribution à la théorie des approximations diophantiques simultanées. Acta Fac. Nat. Univ. Carol., Prague no. **188**, 27 pp. (1948). (French. Czech summary)

The author's main result is as follows. Let s be a positive integer greater than 1. Let $\omega(x)$ be positive and continuous and $\omega^s(x) x^{s+1}$ monotonic for $x \geq 1$. Suppose that $\int_1^\infty \omega^s(x) x^s dx$ converges and $\int_1^\infty \omega^{s-1}(x) x^{s-1} dx$ diverges. Let $\tau(x) \geq x$ be defined for $x \geq 1$, with $\tau(x)/x \to \infty$ as $x \to \infty$. Then there exists a real number Θ_s such that, for almost all real numbers $\Theta_1, \cdots, \Theta_{s-1}$, the system $(\Theta_1, \cdots, \Theta_s)$ admits simultaneous Diophantine approximation with the function $\omega(x)$, but does not admit it with the function $\omega(\tau(x))$. This result is an extension of one due to Jarník [cf. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, chapter 5, theorem 13], which asserted the existence of a system $(\Theta_1, \cdots, \Theta_s)$ with these properties. The construction of Θ_s is effected by a continued fraction, and the set-theoretic part of the proof is less complicated than was the case with Jarník's original proof. *H. Davenport* (London).

Obreškov, N. On Diophantine approximations of linear forms for positive values of the variables. Doklady Akad. Nauk SSSR (N.S.) **73**, 21–24 (1950). (Russian)

The author proves the following simple but elegant variation on a well known result on Diophantine approximation. Let $\omega_1, \cdots, \omega_k$ be real numbers, and n a positive integer. Then there exist integers $x_1, \cdots, x_k$ (not all zero) and y, such that $0 \leq x_i \leq n$ and (*) $|\omega_1 x_1 + \cdots + \omega_k x_k + y| \leq N^{-1}$, where $N = kn + 1$. The proof is by Dirichlet's principle. Give $x_1, \cdots, x_k$ all sets of values satisfying $0 \leq x_i \leq n$, subject to the condition that if $x_i > 0$ then $x_1 = \cdots = x_{i-1} = n$. There are N such sets of values, and if they are arranged in lexicographical order, then $x_i' - x_i \geq 0$ for $i = 1, \cdots, k$ if the set x_i precedes the set x_i'. Considering the N values of $\omega_1 x_1 + \cdots + \omega_k x_k$ (mod 1), it is plain that there must be two sets x_i and x_i' for which the corresponding values of the linear form differ (mod 1) by at most $1/N$. This gives the result. The author further proves that the sign of equality in (*) is necessary if and only if (**) $\omega_1 = \cdots = \omega_k = \lambda/N$ (mod 1), for some integer λ relatively prime to N. He gives an extension to m linear forms $\varphi_i = a_{i1} x_1 + \cdots + a_{ik} x_k$, $i = 1, 2, \cdots, m$, the result being that there exist integers $x_1, \cdots, x_k$ (not all zero) and $y_1, \cdots, y_m$ such that $0 \leq x_i \leq n$ and $|\varphi_i + y_i| \leq N^{-1/m}$ for $i = 1, \cdots, m$. The previous argument extends at once to this case, on considering points $(\varphi_1, \cdots, \varphi_m)$ whose coordinates are treated mod 1. Another extension of the first result is given to the case when the inequalities $0 \leq x_i \leq n$ are replaced by $0 \leq x_i \leq n_i$, and N is taken to be $n_1 + \cdots + n_k + 1$. The author states that the cases of equality are again those given by (**). This does not seem obvious to the reviewer, as the proof given in the previous case does not now apply. *H. Davenport* (London).

ANALYSIS

Mikolás, Miklós. Sur un produit infini. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 68–72 (1950).

Generalizing the formula of Wallis,

$$(2^2/1 \cdot 3) \cdot (4^2/3 \cdot 5) \cdots = \pi/2,$$

the author investigates products of the type

(*) $$P = \prod_{r=0}^{\infty} \left(\frac{a_r + a_{r+1} + \cdots + a_{r+p}}{p+1} \right)^{p+1} \Big/ a_r \cdot a_{r+1} \cdots a_{r+p}, \qquad p \geq 1,$$

where $\{a_r\}$ is an increasing sequence of positive numbers. It is shown that the infinite product (*) converges if and only if the series $\sum_{r=0}^{\infty} ([a_{r+p}/a_r] - 1)^2$ converges. In case $\{a_r\}$ is an arithmetic progression, with $a_r = \kappa r + \lambda$, $r = 0, 1, \cdots, (\kappa, \lambda) = 1$, the author obtains two expressions, according as p is even or odd, giving the value of P in closed form; the formula of Wallis is included as a special case. We note that one might extend the investigation of the present paper by replacing the arithmetic and geometric means in (*) by means of other orders. *E. F. Beckenbach* (Los Angeles, Calif.).

Bertolini, Fernando. Un criterio generale di convergenza per integrali impropri. Applicazioni. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **2**, 125–138 (1950).

The functions $\alpha(x)$, $\beta(y)$ are nondecreasing for nonnegative x and y, and $\gamma(x, y) = \alpha(x)\beta(y)$, $f(x, y)$ is real or complex for (x, y) on the first quadrant and summable on every finite region, $\tau(x, y)$ is defined on the first quadrant as BV in the sense of Vitali, continuous and AC on all intervals on which it is defined. Then $f(x, y)\tau(x, y)$ is summable on every finite region on the first quadrant. Let $A(a_1, a_2)$, $B(b_1, b_2)$ be the ends of a diagonal of a rectangle I, $0 < a_1 < a_2$, $0 < b_1 < b_2$. The functions $s(I) = s(A, B)$, $\sigma(I) = \sigma(A, B)$ are, respectively, (1) $\int_I f(x, y) d\gamma(x, y)$, $\int_I f(x, y)\tau(x, y) d\gamma(x, y)$. The function $f(x, y; P)$ is defined for P on a region C and has the same properties as $f(x, y)$ for each fixed P; $\tau(x, y; Q)$ is defined relative to $\tau(x, y)$ for Q on a region D. The functions $s(I, P)$, $\sigma(I, P, Q)$ are obtained by placing $f(x, y; P)$, $\sigma(x, y; Q)$ in (1). The behaviours of the functions in (1) and of the functions $s(I, P)$, $\sigma(I, P, Q)$ as the points A and B vary are then listed. Let $\tau_{hk}(Q)$ be a double sequence of functions on D for all fixed Q on D, $h \geq 0$, $k \geq 0$, and BV in the sense of Vitali with respect to h, k. The function $f_{hk}(P)$

is defined on C, $h=0, 1, 2, \cdots, k=0, 1, 2, \cdots; \alpha(x)=[x]$, $\beta(y)=[y], x, y \geqq 0, \alpha(x)=\beta(y)=-1, x, y<0$. For $h \leqq x < h+1$, $k \leqq y < k+1$, $\tau(x, y; Q)=\tau_{hk}(Q)$, $f(x, y; P)=f_{hk}(P)$. Two theorems are then proved, one of which is: If the series $\sum_{hk} f_{hk}(P)$, $\sum_h f_{hk}(P)$, $\sum_k f_{hk}(P)$ converge uniformly in C (summation by rectangles in the first sum), and if $\tau_{hk}(Q)$ is BV in the sense of Vitali with respect to (h, k) uniformly in Q on D, then the series $\sum_{hk} f_{hk}(P)\tau_{hk}(Q)$, $\sum_h f_{hk}(P)\tau_{hk}(Q)$, $\sum_k f_{hk}(P)\tau_{hk}(Q)$ converge uniformly in P on C and Q on D. These general theorems are used to obtain the following and other similar results on infinite integrals. The function $f(\theta, z)$ is BV in the sense of Vitali on $-\pi \leqq \theta \leqq \pi$, $-\infty < z < \infty$, and tends to zero as $|z| \to 0$, with $f(-\pi, z)=f(\pi, z)$. The function $v_h(z)=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{ih\theta} f(\theta, z) d\theta$ is BV on $(-\infty, \infty)$, and the Fourier series $\sum e^{ih\theta} v_h(z)$ converges to $f(\theta, z)$ in all intervals of continuity in θ of $f(\theta, z)$. Given $w_h(\alpha)=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\alpha z} v_h(z) dz$, we have $v_h(z)=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\alpha z} w_h(\alpha) d\alpha$, uniformly in all intervals of continuity of $v_h(z)$. It can be concluded that

$$(2\pi)^{\frac{1}{2}} f(\theta, z)=\sum_{-\infty}^{\infty} h \int_{-\infty}^{\infty} e^{i(h\theta + \alpha z)} w_h(\alpha) d\alpha.$$

R. L. *Jeffery* (Kingston, Ont.).

Rosenbaum, R. A. Sub-additive functions. Duke Math. J. 17, 227–247 (1950).

Let $f(p)$ be a measurable real-valued function defined on a semi-module Σ_n in E_n, that is, on a measurable set Σ_n of points in Euclidean space of n dimensions such that if p and p' are points of Σ_n, then so is $p+p'$. If $f(p)$ satisfies $f(p+p') \leqq f(p) + f(p')$ for all p, p' in Σ_n, then $f(p)$ is said to be a sub-additive function. Sub-additive functions occur, for instance, in work on certain classes of weight factors, in the theory of moduli of continuity, and in the uniqueness theory of differential equations; they are of fundamental importance in the study of semi-groups and in the theory of convex bodies. Sub-additive functions in E_1 have been studied extensively by Hille [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948; these Rev. 9, 594].

In the first part of the paper under review the author gives several examples of sub-additive functions, develops their elementary properties, and points out relations between functions of this class and functions of certain other classes. Thus $3+\sin x$ is sub-additive in E_1; and any homogeneous sub-additive function necessarily is convex. In the second part he studies sub-additive functions which might admit the functional values $\pm \infty$. It is shown, for example, that if the sub-additive function $f(p)$ is defined in the entire E_n, and $f(p) \neq +\infty$, then either $f(p) \neq -\infty$ or $f(p) \equiv -\infty$. Next, he studies boundedness and rate of growth properties of sub-additive functions, and finally continuity and differentiability properties of these functions. While they have nearly the same boundedness properties as do additive functions, their continuity and differentiability properties are much weaker. Thus a finite measurable additive function is necessarily linear; but there are finite measurable sub-additive functions which are discontinuous everywhere.

E. F. *Beckenbach* (Los Angeles, Calif.).

Turán, P. On the theory of the mechanical quadrature. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 30–37 (1950).

The following quadrature formula is a special case of a more general formula due to Tschakaloff [C. R. Acad. Bulgare Sci. Math. Nat. 1, no. 2–3, 9–12 (1948); these Rev.

10, 743]:

$$\int_{-1}^{+1} f(x) dx = \sum_{r=1}^{n} [f(x_r)\lambda_r^{(0)} + f'(x_r)\lambda_r^{(1)} + \cdots + f^{(k-1)}(x_r)\lambda_r^{(k-1)}].$$

Here k is an arbitrary integer, $(x_1, \cdots, x_n)$ arbitrary distinct real numbers, $\lambda_r^{(k)}$ real constants depending on k and x_r; this formula holds for all polynomials of degree $kn-1$. The problem is to determine the constants x_r in such a manner that the formula will hold for all polynomials of degree $(k+1)n-1$. If k is even, there is no such system; if k is odd, the x_r must be the zeros of the polynomial $\pi(x)=x^n + \cdots$ minimizing the integral $\int_{-1}^{+1} |\pi(x)|^{k+1} dx$. The resulting formula is a generalization of the quadrature formula of Gauss. Similar questions can be discussed for integrals of the form $\int_{-1}^{+1} f(x) p(x) dx$, where $p(x)$ is a given weight function, generalizing the formula of Jacobi. The special case $p(x)=(1-x^2)^{-1}$ is particularly simple and leads to the following result: Let k be odd; there exist real numbers $\lambda_r^{(k)}$ such that the formula

$$\int_{-1}^{+1} f(x)(1-x^2)^{-\frac{1}{2}} dx = \sum_{r=1}^{n} \sum_{h=0}^{k-1} f^{(h)}\left(\cos \frac{2\nu-1}{2n}\pi\right) \lambda_r^{(h)}$$

holds for all polynomials of degree $(k+1)n-1$.

G. *Szegő* (Stanford University, Calif.).

Erdős, P. Some theorems and remarks on interpolation. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 11–17 (1950).

Let $f(x)$ be defined in $[-1, 1]$ and let $L_n(x)$ be the Lagrange interpolation polynomial of degree $n-1$ coinciding with $f(x)$ at the zeros of the Tchebychev polynomial $T_n(x)=\cos n\varphi$, $x=\cos \varphi$. The following theorems are proved: (a) Let $f(x)$ be continuous in $[-1, 1]$. Then for almost all x, $n^{-1}\sum_{k=1}^{n}|L_k(x)|=o(\log \log n)$; (b) assuming that uniformly in x, $\log \log (|h|^{-1})\{f(x+h)-f(x)\}=o(1)$, we have for almost all x, $n^{-1}\sum_{k=1}^{n}(L_k(x)-f(x))^2 \to 0$.   G. *Szegő*.

✳Steffensen, J. F. Interpolation. 2d ed. Chelsea Publishing Co., New York, N. Y., 1950. ix+248 pp.

Photographic reprint of a book first published in 1927 by The Williams & Wilkins Company, Baltimore, Md.

Calculus

Koschmieder, Lothar. Bemerkung zu einer Formel von Hermite. Jber. Deutsch. Math. Verein. 54, 52–54 (1950). The formula

$$\frac{(x/y)^{n-m+q-p}}{(n-p)!(m-q)!} \frac{\partial^{m+n-p-q}(xy-1)^{m+n}}{\partial x^{n-p}\partial y^{m-q}}$$
$$= \frac{(xy-1)^{p+q}}{(m+p)!(n+q)!} \frac{\partial^{m+n+p+q}(xy-1)^{m+n}}{\partial x^{m+p}\partial y^{n+q}}$$

was stated incompletely and without proof by Hermite. The author proves it. In an appendix, E. Sperner gives a second proof due to M. Gawehn. Both proofs proceed by more or less direct evaluation.   A. *Erdélyi*.

✳Kašanin, Radivoje. Viša matematika. I. [Higher Mathematics. I]. 3d ed. Naučna Knjiga, Belgrade, 1949. vii+847 pp.

✳Kašanin, Radivoje. Viša matematika. II. Knjiga prva. [Higher Mathematics. II. Part 1]. Naučna Knjiga, Belgrade, 1949. viii+624 pp.

A textbook in advanced calculus starting from the beginning. The first volume includes vectors and analytic

geometry, the second infinite series and applications to differential geometry. *W. Feller* (Princeton, N. J.).

*Ollendorff, Franz. **Die Welt der Vektoren. Einführung in Theorie und Anwendung der Vektoren, Tensoren und Operatoren.** Springer-Verlag, Wien, 1950. viii+470 pp. $9.00; bound, $9.50.

This book is as much an encyclopedia as a treatise. It covers a wide range of mathematical and physical topics, beginning in a completely elementary way with an account of Cartesian axes in Euclidean 3-space and ending with an introduction to linear operators in Hilbert space. The work nowhere reaches any great depth; it is not intended to do so. Yet it is equally not intended as a mere catalogue of unrelated ideas. It seeks rather to develop the theory of vectors, tensors, matrices, and linear operators in a unified fashion, and to point the way to the manifold applications of the theory. A bibliography gives a guide to further reading. The nature of the book precludes the giving of more than a bare outline of its scope and contents. Its general plan is to develop the pure mathematics stage by stage, immediately giving applications to physics and to other branches of mathematics.

The first chapter is concerned with the algebra of Cartesian vectors. A vector is defined in the first place as a set of numbers which, under a change of rectangular coordinates (in Euclidean 3-space), transform like the coordinate differentials. An early introduction is made of unit basis-vectors parallel to the coordinate axes, with indices written both above and below as a preparation for the later distinction between contravariant and covariant vectors. The summation convention is used systematically. Sections are then devoted to applications of the vector product to mechanics, and to the use of vector methods in analytical geometry and spherical trigonometry.

The second chapter, concerned with vector fields, deals with such matters as the theorems of Stokes and Gauss, the motion of perfect fluids, the electromagnetic field-equations in free space, irrotational and solenoidal fields, and Huygens' principle. The third chapter introduces nonrectangular reference systems and reciprocal (dual) sets of basis-vectors in Euclidean space of 3 or more dimensions, and has sections on space-lattices, waves in crystals, and triply-periodic functions. Chapter IV is concerned with tensor algebra in Euclidean space, a tensor being defined as a set of numbers transforming like the appropriate product of vectors. Principal directions (Hauptachsen) and invariants (Eigenwerte) associated with a 2-index tensor are introduced, with examples from geometry and mechanics. The next chapter deals with tensor analysis in affine space, with applications to elasticity, viscous fluids, and dielectric polarisation. This is followed by a chapter on Minkowski space and the Lorentz transformation, with applications to electrodynamics and wave mechanics, the last section dealing with the meson field.

Riemannian space makes its appearance in chapter VII, which includes an account of covariant differentiation, curvature, differential operators, various systems of curvilinear coordinates in Euclidean 3-space, classical point-mechanics in Riemannian space, and the general-relativity theory of gravitation. The eighth and last chapter begins with finite-dimensional complex vector space and linear operators therein, with application to electricity. There follows the passage to Hilbert space proper, with an introduction to integral equations, to matrix mechanics, to the theory of

the harmonic oscillator, and so on, the last section dealing with spin-operators.

It is possible that any reader expert in one or more of the subjects dealt with will find things to criticize in the sections of which he has special knowledge. In some places explanations of elementary points may seem unnecessarily laboured, and in others deep matters may seem to be dismissed in a manner too cursory even for an encyclopedic work. Techniques and viewpoints, too, may be the subject of criticism. Yet such things are a matter of opinion rather than of principle, and it seems probable that the general impression will be one of admiration for the wealth of learning displayed by the author in the writing of this book. *H. S. Ruse*.

*Dubnov, Ya. S. **Osnovy vektornogo iscisleniya. Čast I. Vektornaya algebra. Elementy vektornogo analiza. [Foundations of Vector Calculus. Part I. Vector Algebra. Elements of Vector Analysis].** 4th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 368 pp.

The 3d edition was published in 1939. The chapter headings are as follows: Affine correspondences between vectors; scalar multiplication; vector multiplication; vector functions of a scalar variable, scalar fields; differential properties of curves.

*Hague, B. **An Introduction to Vector Analysis for Physicists and Engineers.** 4th ed. Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, N. Y., 1950. ix+122 pp. $1.25.

Theory of Sets, Theory of Functions of Real Variables

Bachmann, Heinz. **Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordnungszahlen.** Vierteljschr. Naturforsch. Ges. Zürich 95, 115–147 (1950).

A function $\varphi(\xi)$ is said to be a normal function of class κ if the following conditions are satisfied: (a) the domain of definition consists of all ordinal numbers ξ with $1 \leq \xi < \omega_\kappa$, and the domain of values is a subset of the domain of definition; (b) if ξ_1, ξ_2 are arbitrary numbers of the domain of definition, then $\xi_1 < \xi_2$ implies $\varphi(\xi_1) < \varphi(\xi_2)$; (c) if ξ is a transfinite limit number $< \omega_\kappa$, then $\varphi(\xi) = \lim_{\xi' < \xi} \varphi(1 + \xi')$. The problem of "distinguished" sequences of ordinal numbers is to assign to every limit number η of the second number class a unique increasing sequence $\eta_0 < \eta_1 < \eta_2 < \cdots$ of type ω such that $\lim_{n < \omega} \eta_n = \eta$. Veblen [Trans. Amer. Math. Soc. 9, 280–292 (1908)] has set up a well-ordered, transfinite sequence $\mathfrak{F}_0$, of type $\omega_1^{\omega_1} + 2$, of normal functions φ_η ($\eta \leq \omega_1^{\omega_1} + 1$) of class one, and, with the aid of these functions, has defined distinguished sequences for all transfinite limit numbers $\eta < E(1)$, where $E(1)$ is a certain limit ordinal of the second number class. The author presents Veblen's method in a form which differs somewhat from the original. He then generalizes this method, employing normal functions of class two in the process, to prolong the sequence $\mathfrak{F}_0$ to a sequence $\mathfrak{F}_1$ of normal functions of class one; and with the aid of $\mathfrak{F}_1$ he defines distinguished sequences for all transfinite limit numbers $\eta < H(1)$, where $H(1)$ is a certain limit ordinal of the second number class $> E(1)$. The meaning of the notion of a complete solution of the problem of distinguished sequences by a further extension of Veblen's method is made precise, and it is shown that a complete

solution of the problem can be obtained if one can construct a prolongation $\mathfrak{F}$, of a certain type $\leq\omega_2$, of the sequence $\mathfrak{F}_1$, satisfying certain natural requirements. The problem is also shown to be equivalent to some others connected with normal functions, including the following. (1) Let V be the set of ordinals ω^ξ ($1\leq\xi<\omega_1$). To assign to every number η of V a unique normal function ψ_η of class one such that η is the first critical number of ψ_η. (The ordinal number ξ is called a critical number of the normal function φ of class κ if $\varphi(\xi)=\xi$. The set of critical numbers of φ is the domain of values of another normal function φ' of class κ.) (2) To assign to an arbitrarily given normal function $\varphi(\xi)$ of class one, whose domain of values is a subset of V, a unique normal function $\Phi(\xi)$, such that $\Phi'(\xi)=\varphi(\xi)$.

F. Bagemihl (Rochester, N. Y.).

Carruth, Philip W. Roots and factors of ordinals. Proc. Amer. Math. Soc. **1**, 470–480 (1950).

If the ordinal numbers $\xi, \beta>0$, α are such that $\xi^\beta=\alpha$, then ξ is said to be a root of α. The author derives necessary and sufficient conditions for a number ξ to be a root of a given number α, and shows that the set of roots of α is closed (i.e., the set contains the least upper bound of each of its nonempty subsets). An ordinal number >1 is called prime if it cannot be written as the product of two ordinals less than itself. Every ordinal number $\alpha>1$ can be expressed as a product of a finite number of prime factors. The author proves that there is a unique decomposition of α into a finite number of prime factors if, and only if, α is not a limit number. Jacobsthal [Math. Ann. **66**, 145–194 (1909), see especially p. 184] and Sieczka [Fund. Math. **5**, 172–176 (1924)] have shown that every $\alpha>1$ has a unique prime factorization if the factorization satisfies certain additional conditions, and this had already been noted by Cantor [Gesammelte Abhandlungen . . . , Springer, Berlin, 1932, pp. 204, 343] for $\alpha<\omega_1$. The author demonstrates that if the following two conditions are satisfied by the prime factorization $\pi_1\pi_2\cdots\pi_k=\alpha$ (where k is a natural number), then it is unique: (1) The number k of prime factors is the smallest number of prime factors into the product of which α can be decomposed; (2) if π_2 is a limit number, then $\gamma\pi_2<\pi_1\pi_2$ for every prime ordinal $\gamma<\pi_1$. The article makes extensive use of Cantor's normal form and Sherman's [Bull. Amer. Math. Soc. **47**, 111–116 (1941); these Rev. **2**, 255] necessary and sufficient condition for one ordinal to be a left factor of another. *F. Bagemihl* (Rochester, N. Y.).

Eyraud, Henri. Le théorème du continu et la divisibilité asymptotique. Ann. Univ. Lyon. Sect. A. (3) **12**, 33–37 (1949).

Results of the author [same Ann. Sect. A. (3) **6**, 33–45 (1943); **9**, 50–54 (1946); C. R. Acad. Sci. Paris **224**, 169–171 (1947); these Rev. **7**, 512; **8**, 448, 320] are proved again or developed further. *F. Bagemihl* (Rochester, N. Y.).

Eyraud, Henri. Les ordinaux des trois premières classes et le théorème du continu. Ann. Univ. Lyon. Sect. A. (3) **12**, 39–50 (1949).

The main part of this paper is a reproduction, with a few changes, of one of the author's notes [same Ann. Sect. A. (3) **11**, 5–8 (1948); these Rev. **10**, 689], and has points in common with other papers of his [ibid. **6**, 33–45 (1943); **7**, 5–13 (1944); **8**, 47–48 (1945); **9**, 27–31 (1946); **10**, 39–42 (1947); these Rev. **7**, 512; **8**, 448; **10**, 689].

F. Bagemihl (Rochester, N. Y.).

Eyraud, Henri. Suites premières de diviseurs. Ann. Univ. Lyon. Sect. A. (3) **12**, 51–52 (1949).

This note is devoted to the construction of a particular example, based on notions defined in the author's article of the second preceding review, and resting on the relation $\aleph=\aleph_1$. *F. Bagemihl* (Rochester, N. Y.).

Eyraud, Henri. Ensembles agrégatifs. Ann. Univ. Lyon. Sect. A. (3) **12**, 53–69 (1949).

The object of this paper is to obtain a one-dimensional continuum C, of power $\aleph_2$, having a subset, of power $\aleph_1$, which is everywhere dense in C. The elements of the simply ordered set C are certain sets of ordinary increasing sequences of natural numbers. The author makes use of notions referred to in the preceding review, and the relations $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ ($\alpha=0, 1$). [Cf., e.g., I. Maximoff, Ann. of Math. (2) **41**, 321–327 (1940); these Rev. **1**, 206.] *F. Bagemihl.*

✶Eyraud, Henri. Leçons sur la théorie des ensembles, les nombres transfinis et le problème du continu. 2d ed. Institut de Mathématiques, Lyon, 1949. ii+90 pp.

This second edition contains less of the classical elementary theory of sets and more of the author's own work [see the papers reviewed and cited above] than the first edition [Lyon, 1947; these Rev. **9**, 230; **10**, 689]. *F. Bagemihl.*

Scorza Dragoni, Giuseppe. Alcune proprietà di struttura per certi insiemi di punti. Ann. Mat. Pura Appl. (4) **28**, 221–229 (1949).

The author gives a detailed discussion of the ϵ-components of sets M of a Euclidean space R_n and, in particular, studies the relation to certain continuity properties of sections of M with linear subspaces. *A. Rosenthal* (Lafayette, Ind.).

Dubreil-Jacotin, Marie-Louise. Quelques propriétés des applications multiformes. C. R. Acad. Sci. Paris **230**, 806–808 (1950).

Let f be a multi-valued mapping of a set E onto a set E', and f^{-1} its inverse. It is proved that: The family $\mathfrak{M}$ of $S\subseteq E$ such that $f^{-1}f(S)=S$ is a complete field of sets; there are minimal S; these minimal S define a partition $\mathfrak{R}$ of E which can also be defined by an equivalence C. If $\mathfrak{M}'$ is the family of $S'\subseteq E'$ such that $F^{-1}F(S')=S'$ (where $F=f^{-1}$), and $\mathfrak{R}'$ the partition defined by the minimal S', and C' the corresponding equivalence relation, then it is shown that $\mathfrak{M}$ and $\mathfrak{M}'$ are in one-to-one correspondence. Since the correspondence preserves inclusion, E/C and E'/C' are in one-to-one correspondence also. The last correspondence generalizes the classical case (where f is single-valued) with $f^{-1}(y)\leftrightarrow y$. Two examples are given to illustrate the above. *H. Tong* (Paris).

Dubreil-Jacotin, Marie-Louise. Applications multiformes et relations d'équivalences. C. R. Acad. Sci. Paris **230**, 906–908 (1950).

The author continues her work on the fundamental properties associated with multi-valued mappings [cf. the preceding review]. If R and R' are two equivalences defined in E and E' respectively, "R' is f-compatible with R" is defined to mean "$xRy\rightarrow x'R'y'$ for every $x'\varepsilon f(x)$ and every $y'\varepsilon f(y)$." This definition generalizes the notion of f-compatibility considered by Bourbaki [Théorie des ensembles . . . , Actual. Sci. Ind., no. 846, Hermann, Paris, 1939, p. 31; these Rev. **3**, 55] with single value f. If R' is f-compatible with R it is readily seen that $C'\subseteq R'$. Let $E^*=E/R$ and a mapping f^* of E^* onto E' be defined by $f^*(b^*)=\bigcup_{b\varepsilon b^*}f(b)$,

where $b^*\epsilon E^*$. It is shown that R' is f-compatible with R if and only if the images of f of classes modulo R are also contained in the classes modulo R'. Since the intersections of a family of equivalences f-compatible with R is an equivalence f-compatible with R, there is a smallest equivalence R_1' f-compatible with R; it is in fact the fundamental equivalence C' corresponding to the mapping f^* of E^* onto E'. It is then shown that $a'R_1'b'$ is equivalent to aRb for every $a\epsilon f^{-1}(a')$ and every $b\epsilon f^{-1}(b')$ is equivalent to the existence of an $a\epsilon f^{-1}(a')$ and a $b\epsilon f^{-1}(b')$ such that aRb. An example of non-f-compatibility is given. *H. Tong.*

Goffman, Casper. On Lebesgue's density theorem. Proc. Amer. Math. Soc. 1, 384–388 (1950).

The density theorem states that the metric density of a measurable linear point set is 0 or 1 except on a set of measure zero. This note considers the extent to which an arbitrary set Z of measure zero can play the role of the exceptional set. The results are: (1) for any Z there is a measurable S such that the density is not defined on Z; (2) if S is measurable and Z_1 is the set on which the density of S exists but is neither 0 nor 1, then Z_1 is of the first category; (3) if Z is an F_σ set of measure zero, then there is a measurable S whose metric density is $\frac{1}{2}$ on Z.

L. W. Cohen (Flushing, N. Y.).

Maharam, Dorothy. Decompositions of measure algebras and spaces. Trans. Amer. Math. Soc. 69, 142–160 (1950).

By a measure algebra (abbreviated m.a.) is meant a Boolean σ-algebra with a σ-finite positive measure. Let (A_n, μ_n), $n \geq 0$, be a sequence of m.a.'s, and let K be a principal ideal in the direct product of (A_0, μ_0) with a m.a. (J, m). Form the direct sum (E^*, μ^*) of the m.a.'s $(K, \mu_0 \times m)$, (A_1, μ_1), (A_2, μ_2), $\cdots$. The elements of A_n, $n \geq 1$, together with the "cylindrical elements" of K based on A_0 generate a σ-subalgebra A^* of E^*. The main theorem of this paper asserts that if (E, μ) is any m.a. and if A is any σ-subalgebra of E, then it is possible to find a sequence of principal ideals A_n ($n \geq 0$) in A, where $A_n \supset A_{n+1}$ for $n > 0$, and to define corresponding measures μ_n, a m.a. (J, m), and a principal ideal K in $A_0 \times J$, such that (E, μ) and the direct sum (E^*, μ^*) described above are isomorphic under an isometry that makes A correspond to A^*. This result is also formulated in two other ways, which relate it to similar algebraic decomposition theorems announced by Nikodým and Gleason. A characterization of the case in which A consists of all elements invariant under some group of automorphisms of (E, μ) is then obtained. The main theorem is used to derive an analogous decomposition of a measure space relative to a given countably generated subfield of measurable sets, assuming that the atom-free part of the measure space is isometric to a finite or infinite interval of the real line. Finally, these results are applied to obtain a decomposition of a measure preserving transformation or flow into ergodic parts. Most of the proofs depend on results in an earlier paper by the author [same Trans. 65, 279–330 (1949); these Rev. 10, 519]. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Schaerf, H. M. On the role of an intersection property in measure theory. I. Portugaliae Math. 8, 95–102 (1949).

If for every two sets A and B, of positive finite measure with respect to a measure m, there exist sets A' and B' of positive measure such that $A' \subset A$, $B' \subset B$, and $A'RB'$, where R is a prescribed binary relation between measurable sets, the measure m is said to have the intersection property

modulo R. Abstracting from the well-known group-theoretic situation, the author obtains some results whose hypotheses involve this intersection property and whose conclusions assert that the measures in question are uniquely determined in one or another sense (e.g., up to absolute continuity equivalence, or up to a multiplicative constant factor).

P. R. Halmos (Chicago, Ill.).

*Luzin, N. N. Teoriya funkciĭ deĭstvitel'nogo peremennogo. Obščaya čast'. [Theory of Functions of a Real Variable. General Part]. 2d ed. Gosudarstvennoe Učebno-Pedagogičeskoe Izdatel'stvo Ministerstva Prosveščeniya SSSR, Moscow, 1948. 318 pp.

This book, according to its introduction, is designed as a textbook in the elementary theory of functions of a real variable, the time not yet being ripe for a complete exposition of the theory of sets and of functions of a real variable. Chapters I and II deal with the set concept, cardinal numbers, and point sets on the line. These chapters leave much to be desired as an exposition for students. The explanations of sets and cardinal numbers are at once verbose and indefinite, while the proofs offered are for the most part mere plausibility arguments. As an example of the inaccuracies encountered, one notes the following assertion, which is misleading at best: All properties of linear sets can be inferred from the facts that the rational numbers are dense on the line and that the intersection of a sequence of nested closed intervals with diameters approaching 0 contains exactly one point (p. 55). Later chapters are much more lucid. Chapter III takes up the Moore-Smith limit for real-valued functions defined on arbitrary completely ordered sets without last element, thus treating simultaneously limits of ordinary sequences and limits of real-valued functions defined on an open interval. In chapter IV, functions and continuity are defined, the usual elementary facts about continuity being set forth in great detail. In chapter V, continuous curves according to the definitions of Cantor and Jordan are studied. Peano's space-filling curve is constructed. Chapter VI, perhaps the most valuable in the book, is devoted to the problem of analytic representation of functions, essentially by limits of polynomials. Uniform and quasi-uniform convergence are studied, Lebesgue's proof of the Weierstrass approximation theorem is given, Čebyšev's earlier work is noted, and a long discussion of the Bernstein polynomials is presented. Two appendices, dealing with Dedekind cuts and with the Baire classification, complete the book. *E. Hewitt* (Seattle, Wash.).

Tolstov, G. P. Integral as a primitive. Doklady Akad. Nauk SSSR (N.S.) 73, 659–662 (1950). (Russian)

The author's first problem is to find from a property of f, defined on $a \leq x \leq b$, corresponding properties that can be demanded of a monotone φ, defined on $\alpha \leq t \leq \beta$, with $\varphi(\alpha) = a$, $\varphi(\beta) = b$, so that

$$(1) \qquad \int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

is satisfied. Theorem 1. If f is finite, summable, and has only a closed set of measure zero of discontinuities, then φ exists such that $\varphi'(t)$ and $f(\varphi(t))\varphi'(t)$ are continuous. Theorem 2. If f is finite and summable, then φ exists with bounded derivative such that $f(\varphi(t))\varphi'(t)$ is everywhere a bounded derivative. Theorem 3. If f is finite and Denjoy integrable in the strict sense, then φ exists with $\varphi'(t)$ everywhere finite such that $f(\varphi(t))\varphi'(t)$ is an everywhere finite derivative. In the proofs, let $F(x) = \int_a^x f(\xi)d\xi$, and let s be

arc length along the graph of F. Choosing $s(t)$ so that $s'(t) = 0$ on an appropriate t-set, $\varphi(t) = x(s(t))$ has the desired properties and $f(\varphi(t))\varphi'(t) = dF(\varphi(t))/dt$. Theorem 4. If $f(x, y)$ is defined and measurable on the strip $a \leq x \leq b$, satisfies a Lipschitz condition in y and, even if only for one value $y = y_1$, is integrable in the strict sense of Denjoy in x, then the differential equation (2) $dy = f(x, y)dx$ has a parametric solution $x = \varphi(t)$, $y = \psi(t)$, $\alpha \leq t \leq \beta$, satisfying the initial conditions $y = y_0$ when $x = x_0$. Moreover, when t approaches the limits α and β, x approaches the limits a and b, and $f(x, \eta) = f(x, y_1) + \int_{y_1}^{\eta} f_y(x, y)dy$; hence, for every continuous $\eta(x)$, $f(x, \eta(x))$ is Denjoy integrable. Iteration gives a continuous solution of $\eta(x) = y_0 + \int_{x_0}^{x}(f(\xi, \eta(\xi))d\xi$. Theorem 3 applied to this integral yields $x = \varphi(t)$ and $y = \eta(\varphi(t))$ as a parametric solution of (2).   *M. M. Day* (Urbana, Ill.).

Hartman, S., and Marczewski, E. On the convergence in measure. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 125–131 (1950).

Let (X, M, μ) be a probability space and, for every point $a = (a_1, \cdots, a_k)$ in the k-space R^k, let L_a be the set of all points $(x_1, \cdots, x_k)$ in R^k with $x_i < a_i$, $i = 1, \cdots, k$. It is well known that the statement A: "A sequence f_n of M-measurable functions defined on X and taking values in R^k is convergent in μ-measure to a function f" implies the statement B: "The distribution function of f_n converges to the distribution function F of f at each continuity point a of F." In this paper the stronger result is obtained that statement A is equivalent with the following statement C: "The μ-measure of the symmetric difference of $f_n^{-1}(L_a)$ and $f^{-1}(L_a)$ converges to 0 at each continuity point a of F." In fact, generalizations of this result to functions taking values in a metric space Y are obtained, and the following conclusions are derived: (a) The independence of the nth terms $(n = 1, 2, \cdots)$ of two sequences of such functions implies the independence of their limits in measure; (b) if Y is of the power of the continuum, the limit in measure of a sequence of functions containing independent pairs of elements with arbitrarily large subscripts is constant almost everywhere.   *H. M. Schaerf* (St. Louis, Mo.).

Areškin, G. Ya. On convergence in length of curves and on curvilinear Lebesgue integrals. Doklady Akad. Nauk SSSR (N.S.) 72, 821–824 (1950). (Russian)

Let C_k, $k = 0, 1, \cdots$, be rectifiable Jordan curves defined by parametric equations $x^i = \varphi_k^i(t)$, $i = 1, 2, 3$; $t \epsilon I$, a closed interval. If L_k is the length of C_k, reparametrizing C_k by arc length yields a one-to-one correspondence between the Lebesgue measurable subsets of $0 \leq s \leq L_k$ and a Borel field K_k of subsets of I. Letting K_ω be the intersection of all K_k, Lebesgue measure on $0 \leq s \leq L_k$ defines, for each k, a length function $L_k(T)$, $T \epsilon K_\omega$, and K_ω contains B, the family of Borel sets in I. Now assume that φ_k^i converges uniformly to φ_0^i for each i. If K is a field of sets contained in K_ω, define $C_k \to C_0(L, K)$ to mean $L_k(T) \to L_0(T)$ for every $T \epsilon K$. An example of a Borel set T and curves C_k is given to show that $L_k(I) \to L_0(I)$ need not imply $C_k \to C_0(L, B)$. Theorem. For $C_k \to C_0(L, K_\omega)$ it is necessary and sufficient that (1) $L_k(I) \to L_0(I)$ and (2) the set functions L_k, $k = 1, 2, \cdots$, be equicontinuous on K^0_{br}, where K^0 is the smallest field containing all subintervals of I. Equicontinuity means here that if $\{T_n\}$ shrinks to the null set, then $\lim_n L_k(T_n) = 0$ uniformly in k. Under the hypothesis that $C_k \to C_0(L, K_\omega)$ a set of additional conditions is stated sufficient for convergence of line integrals $\int_{Q_k} f_k ds$ to $\int_{Q_0} f_0 ds$, where the inte-

grals are interpreted in the sense of Lebesgue-Stieltjes, f_k is a suitably restricted function on C_k, Q_k is the image on C_k of a set $T_k \epsilon K_\omega$, and $T_k \to T_0$.   *M. M. Day.*

Mathis, H. F. A theorem on the extension of a rectangular matrix of continuous functions to become a nonsingular square matrix. Proc. Amer. Math. Soc. 1, 344–345 (1950).

L'autore estende alle funzioni continue di più variabili un teorema dato da Bliss per quelle di una variabile [Trans. Amer. Math. Soc. 19, 305–314 (1918), p. 312]. Precisamente, se la matrice

$$A = \begin{Vmatrix} a_{1,1}(t_1, \cdots, t_n) \cdots & a_{1,q}(t_1, \cdots, t_n) \\ \cdots & \cdots & \cdots \\ a_{p,1}(t_1, \cdots, t_n) \cdots & a_{p,q}(t_1, \cdots, t_n) \end{Vmatrix} \quad (p < q)$$

dove le $a_{i,j}$ son funzioni continue in un insieme R omemomorfo ad una n-cella chiusa, ha sempre caratteristica massima in R, essa si può pensare come sub-matrice di una matrice M, d'ordine q, costituita tutta da funzioni continue e non singolare in tutto R. Gli elementi di M che non compaiono in A possono allora supporsi essere polinomi. La dimostrazione si riconduce al caso che R sia un ipercubo e procede per induzione.   *G. Scorza-Dragoni* (Padova).

Myškis, A. D. The complete differential at a generalized boundary point. Mat. Sbornik N.S. 26(68), 345–350 (1950). (Russian)

Let R be a bounded open set in Cartesian n-space E_n, let $r(x, y)$ be the ordinary distance of the points x, y and let $r^*(x, y)$ be, for x, y in R, the minimal length of arcs in R joining the points x, y. Further let $r(x, Y)$ and $r^*(x, Y)$ denote the (minimal) r and r^* distances from the point x to the point-set Y. In accordance with one of the author's earlier definitions [Mat. Sbornik N.S. 25(67), 387–414 (1949); these Rev. 11, 382], a generalized boundary point of R is a class M of nonempty open subsets G of R such that the intersection of any two members G_1, G_2 of M contains as subset a member G of M, while the intersection of all members of M is empty; M is termed "nicely approachable" in case of finiteness of the number $\inf_{G \epsilon M} \sup_{x \epsilon G} r^*(x)/r(x)$, where $r^*(x) = \sup_{G \epsilon M} r^*(x, G)$, $r(x) = \sup_{G \epsilon M} r(x, G)$. (Here $r^*(x)$ and $r(x)$ can be regarded as distances from x to M.) A function $f(x)$ defined for x in some $G_1 M$ is termed continuous at M and we write $f(M) = l$, if given $\epsilon > 0$ there exists $G \epsilon M$ such that $|f(x) - l| < \epsilon$ for all $x \epsilon G$. The plane $y = \sum a_i x_i + b$, where x_i is the ith component of x, touches $y = f(x)$ at M, if $f(x) = \sum a_i x_i + b + \epsilon(x)r(x)$, where $\epsilon(M) = 0$. With these definitions the author's main theorem states that if M is nicely approachable, if further $f_i(M) = a_i$, $i = 1, \cdots, n$, where $f_i(x)$ is the partial derivative of $f(x)$ in x_i, and if $\varphi(M) = b$, where $\varphi = f - \sum a_i x_i$, then the plane $y = \sum a_i x_i + b$ touches $y = f(x)$ at M. Conversely, if M is not nicely approachable, this is false for some f, at any rate when M has a countable basis.   *L. C. Young.*

Besicovitch, A. S. Parametric surfaces. Bull. Amer. Math. Soc. 56, 288–296 (1950).

This is the text of an expository lecture on surface area theory, presented by invitation at the annual meeting of the American Mathematical Society in December, 1948. The author presents in detail his objections to the Lebesgue definition of surface area, and outlines his own approach.   *T. Radó* (Columbus, Ohio).

Mickle, Earl J. Some examples in surface area theory. Rivista Mat. Univ. Parma 1, 197–206 (1950).

I. There exists a Fréchet surface S of finite Lebesgue area having no representation $T: x^i = x^i(u, v)$, $i = 1, 2, 3$, $(u, v)\epsilon Q$ (Q, unit square), with $x^i(u, v)$ functions of bounded variation in the sense of Tonelli. This example proves the necessity of dropping such a condition in the representation theorems recently proved for all surfaces of finite Lebesgue area [Cesari, Amer. J. Math. 72, 335–346 (1950); these Rev. 11, 588].

II. Given a representation T of a surface S, for each interior point $w = (u, v)\epsilon Q$, let us consider the two points $w' = (u + h, v)$, $w'' = (u, v + h)$ and the area α of the triangle inscribed in S with vertices $T(w)$, $T(w')$, $T(w'')$. Let

$$D(u, v; h) = \alpha/h^2, \quad I(h) = \iint_Q D(u, v; h) du\, dv,$$

$$A_*(T) = \lim \inf I(h), \quad A^*(T) = \lim \sup I(h)$$

as $h \to 0$. The expressions $A_*(T)$, $A^*(T)$ have a remarkable importance in the study of Geöcze's problem. The author proves that there exists a surface S of zero Lebesgue area and a representation T of S such that $A_*(T) = +\infty$.

III. L. C. Young [Fund. Math. 35, 275–302 (1948); these Rev. 10, 520] proposed a definition of "area" $A_y(T)$ for any continuous mapping T as the two-dimensional Hausdorff measure of the set $T(Q)$, where each point $x = (x^1, x^2, x^3)$ of $T(Q)$ has a multiplicity equal to the number of components of the inverse set $T^{-1}(x) \subset Q$. The author proves that such an "area" is not monotone, that is, there exists a mapping T defined on Q such that, if we call T' the submapping defined by T on a subrectangle $R \subset Q$, we have $A_y(T') > A_y(T)$ instead of $A_y(T') \le A_y(T)$. *L. Cesari.*

Theory of Functions of Complex Variables

Brödel, Walter. Über die Nullstellen der Weierstrass-schen $\wp$-Funktion. J. Reine Angew. Math. 187, 189–192 (1950).

Über die Lage der Nullstellen von $\wp(u \mid \omega, \omega')$ ist bekannt: Im Fundamentalparallelogramm $(u = \theta \cdot 2\omega + \theta' \cdot 2\omega'$, mit $0 \le \theta < 1$, $0 \le \theta' \le 1)$ liegen genau zwei Nullstellen. Diese sind im allgemeinen verschieden und fallen dann und nur dann zusammen, wenn das Fundamentalparallelogramm ein Quadrat ist, oder wenn jedenfalls das Periodengitter durch ein Quadrat erzeugt werden kann (lemniskatischer Fall). Die alsdann doppelt zu zählende Nullstelle liegt im Mittelpunkte des Fundamentalquadrats. Im allgemeinen Falle sind den beiden Nullstellen zwei Wertepaare (θ, θ') und $(1 - \theta, 1 - \theta')$ zugeordnet, wobei auftretende Einsen durch Nullen zu ersetzen sind. Der Verf. beweist eine von Hasse ausgesprochene Vermutung, welche besagt, dass jedes Wertsystem θ, θ' $(0 \le \theta < 1$, $0 \le \theta' < 1)$, unter Ausschluss von $\theta = \theta' = 0$) bei geeigneter Wahl des Periodenverhältnisses eine Nullstelle der $\wp$-Funktion in der Form $u = \theta \cdot 2\omega + \theta' \cdot 2\omega'$ liefern kann. Er setzt $2\omega = 1$, $2\omega' = \tau = s + it$ $(t = \Im \tau > 0)$. Die unendlich vielen zu den Grundperioden $(1, \tau)$ gehörigen Nullstellen V_n werden bestimmt durch

$$V_n(\tau) = \int_0^\infty \tfrac{1}{2}[(s - e_1)(s - e_2)(s - e_3)]^{-\frac{1}{2}} ds$$

(Integrationsweg auf der zweiblättrigen bei e_1, e_2, e_3, und ∞ verzweigten Riemannschen Fläche), $e_1 = \wp(\frac{1}{2} \mid \frac{1}{2}, \frac{1}{2}\tau)$,

$e_2 = \wp(\frac{1}{2} + \frac{1}{2}\tau \mid \frac{1}{2}, \frac{1}{2}\tau)$, und $e_3 = \wp(\frac{1}{2}\tau \mid \frac{1}{2}, \frac{1}{2}\tau)$. Die Zahlen θ und θ' sind bestimmt durch $\Re v = x = \theta + \theta's$; $\Im v = y = \theta' t$. Der Verf. betrachtet die Abbildung der τ-Ebene auf eine Ebene mit den kartesischen Koordinaten θ, θ', Er zeigt erst, dass das Innere des τ-Gebietes $(|\tau| \ge 1$, $0 \le s \le \frac{1}{2})$ auf das schlichte Innere des Dreiecks $\theta = \frac{1}{2}$, $\theta = \theta'$, $2\theta + \theta' = 1$ abgebildet wird, und in ähnlicher Weise ergeben sich die Bilder für die fünf weiteren Kreisbogendreiecke, die insgesamt das Kreisbogendreieck $0 \le \delta \le 1$, $|\tau - \frac{1}{2}| \ge \frac{1}{2}$ bedecken und durch Einzeichnen der Geraden $s = \frac{1}{2}$ sowie der Kreisbogen $|\tau| = 1$ und $|\tau - 1| = 1$ entstehen. Zum Beweise des Hasseschen Vermutung bedarf es nur noch einer Bemerkung über die Gerade $\theta' = 0$. *S. C. van Veen* (Delft).

Jacobsthal, Ernst. Sur l'inversion d'une série de puissances. I–VI. Norske Vid. Selsk. Forh., Trondheim 21, 13–17, 18–21 (1948); 124–129, 130–135, 136–140, 141–144 (1949).

Dans une note [Rend. Circ. Mat. Palermo 54, 42–46 (1930)], Ward a donné explicitement sous forme de déterminants les coefficients du développement taylorien autour de l'origine de la fonction inverse, s'annulant à l'origine, de la fonction holomorphe $y = f(x) = a_1 x + a_2 x^2 + \cdots$ donnée par les coefficients a_1, a_2, $\cdots$. Il employait à cet effet la formule de Lagrange. L'auteur retrouve le résultat (notes I et II) en utilisant la formule de Cauchy

$$x = (2i\pi)^{-1} \int [uf'(u)/(f(u) - y)] du,$$

l'intégrale étant prise sur un contour convenable. Il donne d'autre part les expressions explicites des coefficients des formules d'inversion lorsque $f(x)$ est définie par un développement autour d'un pôle simple à l'origine ou par un développement autour du point à l'infini qui est zéro simple ou pôle simple. Il en déduit notamment une expression des nombres de Bernoulli qu'il compare à une formule de Frobenius (note II). Il montre (note III) comment les formules se simplifient dans le cas des fonctions impaires et applique de nouveau le résultat aux nombres de Bernoulli. Il examine ensuite le cas où le zéro ou pôle à l'origine ou à l'infini est d'ordre q supérieur à un (notes IV et V) et montre que dans ce cas encore il y a des simplifications si la fonction $f(x)$ est paire et $q = 2$ (note VI). *G. Valiron* (Paris).

Azpeitia, A. G. Note on power series. Gaceta Mat. (1) 2, 15–17 (1950). (Spanish)

The author's theorem and his proof of it are contained in Pólya and Szegö [Aufgaben und Lehrsätze aus der Analysis, v. 2, Springer, Berlin, 1925, p. 169, solution to problem 8]. It is also given at greater length by Dietrich and Rosenthal [Bull. Amer. Math. Soc. 55, 954–956 (1949); these Rev. 11, 331]. *R. M. Redheffer* (Los Angeles, Calif.).

Pi Calleja, Pedro. On determination of the singularities of Taylor's series by means of the argument of its coefficients. Revista Unión Mat. Argentina 14, 226–231 (1950). (Spanish)

(i) A critical and historical analysis of Fabry's theorem [see, e.g., Dienes, The Taylor Series . . . , Oxford, 1931, p. 377] and corollaries with special reference to simpler but less general theorems. (ii) If $\sum_0^\infty a_n z^n$ has radius of convergence unity and $\lim |\Delta^n a_1|^{1/n} = 0$, then $z = 1$ is its only singularity.

A. J. Macintyre (Aberdeen).

Wilson, R. Analogues for integral functions of certain theorems on power series. Quart. J. Math., Oxford Ser. (2) 1, 211–214 (1950).

By using the generalized Laplace transform connecting $\sum a_n z^{-n-1}$ with $\sum a_n z^n / \Gamma(\sigma+n\sigma)$ [see A. J. Macintyre, Proc. London Math. Soc. (2) 45, 1–20 (1938)], the author derives three theorems on entire functions from three theorems on singular points of power series. The simplest is that if $f(z)=\sum_0^\infty c_n z^n$ is an entire function of order ρ and $\lim n^{1/\rho} c_{n+1}/c_n = ae^{i\alpha}$ exists, then $f(z)$ is of type a^ρ/ρ and $e^{-i\alpha}$ is a direction of strongest growth of $f(z)$. This corresponds to one of Fabry's theorems on power series and the other two theorems correspond to successively stronger generalizations.   *R. P. Boas, Jr.* (Evanston, Ill.).

Milloux, Henri. Sur les directions de Borel des fonctions entières et de leurs dérivées. C. R. Acad. Sci. Paris 231, 402–403 (1950).

The domain D of the z-plane is said to be a domain of repletion of valency n of $f(z)$ (domaine de remplissage d'indice n), if $f(z)=a$ has at least n solutions in D for every a except perhaps for a set of a which can be enclosed in two circles of radius ϵ^{-n} on the Riemann sphere. Theorem. Let $f(z)$ be an integral function of finite positive order. There is at least one direction V such that every angle bisected by V contains an infinity of circles of $f(z)$ and of every derivative of $f(z)$. The centres of these circles tend to infinity. For every $f^{(k)}(z)$ the valency of the circles is so great that V is a Borel direction of $f(z)$ and all its derivatives. The theorem is also true for functions of zero order whose characteristic function $T(r)$ satisfies $\lim \sup T(r)(\log r)^{-\alpha} > 0$ $(\alpha > \frac{1}{2})$. The author states that the proof is based on the study of functions $h(z)$ regular in a domain D and such that $h'(z)$ has a large number of zeros in an interior domain. Such a function is either very nearly constant or it has D as domain of repletion.
                    W. H. J. Fuchs (Ithaca, N. Y.).

Riesz, Marcel. Remarque sur les fonctions holomorphes. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 53–56 (1950).

The author establishes an infinitesimal inequality for functions analytic in the closed unit circle. This inequality leads to an inequality of Carlson [Ark. Mat. Astr. Fys. 29B, no. 11 (1943); these Rev. 6, 205] which in turn implies the classical results of Fejér and F. Riesz [Math. Z. 11, 305–314 (1921)] and Gabriel [Proc. London Math. Soc. (2) 28, 121–127 (1928)].   *M. Heins* (Providence, R. I.).

Bohr, Harald. Zur Theorie der Dirichletschen Reihen. Math. Z. 52, 709–722 (1950).

The object of this note is the construction of an ordinary Dirichlet series $\sum a_n n^{-s}$ having the following properties. Its abscissa of convergence is 0 and its abscissa of (C, α) summability is $-\alpha$ for all $\alpha > 0$. The function $f(s)$ defined by the series is an entire function whose Lindelöf μ-function is $\mu(\sigma)=0$ for $\sigma>1$, $\mu(\sigma)=1-\sigma$ for $\sigma\leq1$. It is known that if the abscissa of (C, α) summability of a Dirichlet series is $-\alpha$ then $-\sigma \leq \mu(\sigma) \leq 1-\sigma$. In an earlier note [see Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 6 (1949); these Rev. 11, 168] the author showed the existence of a Dirichlet series for which $\mu(\sigma)=-\sigma$. In the present note he shows by an ingenious construction that the other extreme $\mu(\sigma)=1-\sigma$ may also be attained. There is a disturbing misprint in the definition of $f(s)$ on p. 715, where in three places tm must

be replaced by t_m. Let $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ be summable (C, α) and (C, β) respectively for $\sigma>0$, where $0 \leq \alpha \leq \beta$. Then the product series is convergent for $\sigma > \frac{1}{2}(\alpha+\beta+1)$ if $\alpha \leq \beta \leq \alpha+1$ and for $\sigma > \beta$ if $\beta \geq \alpha+1$. With the aid of the series constructed in the present note he shows that these results are the best possible for all values of α and β.
                    E. Hille (New Haven, Conn.).

Hirschman, I. I., Jr., and Jenkins, J. A. On lacunary Dirichlet series. Proc. Amer. Math. Soc. 1, 512–517 (1950).

Let $f(s)=\sum a_k e^{-\lambda_k s}$ $(0<\lambda_k \uparrow \infty)$ be a Dirichlet series with abscissa of convergence $\gamma_c < \infty$ and abscissa of analyticity $\gamma_a > -\infty$. The authors prove a general theorem asserting that if the λ_k's are lacunary, then, unless $f(s)=0$, $f(\sigma+i\tau_a)$ cannot tend to zero too rapidly for $\sigma \downarrow \gamma_a$. The following is a typical example of their results: let $\nu<1$ be the exponent of convergence of the λ_k's and suppose that

$$f(s_a+\sigma)=O[\exp((-1/\sigma)^\mu)],$$

where $s_a=\gamma_a+i\tau_a$. Then, if $\mu>\nu/(1-\nu)$, $f(s)=0$.
                    S. Agmon (Houston, Tex.).

Warschawski, S. E. On conformal mapping of nearly circular regions. Proc. Amer. Math. Soc. 1, 562–574 (1950).

Let R be a given simply connected region in the w-plane whose boundary is given by $\rho=\rho(\phi)$ in polar coordinates where $1<\rho(\phi)<1+\epsilon$, $\epsilon>0$, and let $w=f(z)$ be the function which maps the interior of the unit circle $|z|<1$ conformally on R subject to the normalization $f(0)=0$ and $f'(0)>0$. Also let $\mathfrak{M}_p\{f(z)-z\} = \{(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})-re^{i\theta}|^p d\theta\}^{1/p}$. The author derives the following bounds for the integral means of order $p>0$: $\mathfrak{M}_p\{f(z)-z\} \leq \mathfrak{M}_p\{f(z)/z-1\} \leq (1+A_p)e^\epsilon$, where A_p is the Riesz constant. Under a Lipschitz condition on $\rho(\phi)$ he also shows that

$$\mathfrak{M}_p\{zf'(z)/f(z)-1\} \leq \epsilon(1+A_p)/(1-\epsilon A_p)$$

if $\epsilon A_p<1$. In addition it is proved that

$$\mathfrak{M}_p\{f'(z)-1\} \leq 2\epsilon e^\epsilon(1+A_p)/(1-\epsilon A_p)$$

if $\epsilon A_p<1$; and under a Lipschitz condition on $\rho'(\phi)/\rho(\phi)$ it is shown that $\mathfrak{M}_p\{\phi''(\theta)\} \leq AM_p\epsilon$, where A and M_p are constants, and $\theta=\arg z$, $\phi=\arg w$, and $\phi(\theta)$ is defined by the point-to-point correspondence of the boundaries of R and the unit circle under the given mapping. Using these theorems the author proves that $|f(z)-z| \leq K\epsilon$, where K is an absolute constant, and that $A^{-1}(1+\epsilon)^{-1} \leq \phi'(\theta) \leq A$, and $|f'(z)-1| < 2B(1+\epsilon)\epsilon+\epsilon$, where A and B are constants. Some of these theorems were stated without proof and used in a previous paper by the author [Quart. Appl. Math. 3, 12–28 (1945); these Rev. 6, 207].   *C. Saltzer.*

Wittich, Hans. Konvergenzbetrachtung zum Abbildungsverfahren von Theodorsen-Garrick. Math. Ann. 122, 6–13 (1950).

The calculation by the Theodorsen-Garrick method of the function which maps the interior of a circle conformally on a given star-shaped region whose boundary in polar coordinates is $P=P(\theta)$ depends on the approximation by iteration of the solution of the Theodorsen-Garrick integral equation,

$$g(\phi)=L[g(\phi)]=(2\pi)^{-1}\int_0^{2\pi} f[\alpha+g(\alpha)]\cot\tfrac{1}{2}(\alpha-\phi)d\alpha,$$

where $f(\phi)=\log P(\phi)/R$ and R is constant. The author's principal result is that, under certain Hölder conditions on $f'(\phi)$ and the first approximation $g_0'(\phi)$, the sequence

$\{g_n(\phi)\}_{n=0}^{\infty}$ defined by $g_0(\phi)$ and the relation $g_{n+1}(\phi) = L[g_n(\phi)]$ converges uniformly. [The uniform convergence of this sequence was established under much less restrictive conditions by Warschawski, Quart. Appl. Math. **3**, 12–28 (1945); these Rev. **6**, 207.] *C. Saltzer* (Chelmsford).

✶ **Julia, Gaston. Quelques applications fonctionnelles de la topologie.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 291–306, Rome, 1943.

This paper is an historical summary of classical results on the theory of functions of a single complex variable with contributions by the author from 1917 to the date of the paper. Three chapters are headed: (I) inégalités, "majorantes," en théorie des fonctions; (II) quelques problèmes de représentation conforme; (III) équations fonctionnelles considérées comme relations sur une surface de Riemann.

In (I) reference is made to (1) the principal of the maximum modulus (Cauchy) and extensions of Phragmén and Lindelöf, (2) the lemma of Schwarz with applications by the author, (3) extensions of Lindelöf to an arbitrary domain d mapped conformally onto a Riemann domain D "portée" by a Riemann domain D_1, (4) the generalization of Littlewood using functions subordinated one to the other, (5) "majoration" in domains touching the frontier of d, including the author's results of 1917 enlarged by Nevanlinna, Wolff, and Carathéodory, (6) the ultimate extensions in the thesis of Beurling [Uppsala, 1933].

In (II) reference is made to the canonical conformal mappings of Riemann, Poincaré, Schottky, Koebe, and the problem of de La Vallée Poussin [Ann. Sci. École Norm. Sup. (3) **47**, 267–309 (1930)], more particularly "la représentation conforme canonique d'une aire $d(z)$ de genre P sur une aire $D(Z)$ limitée par des cassiniennes appartenant à un polynome $P(Z)$ de degré P." The author summarizes his contributions and refers to several notes and memoires.

In (III) the author shows the use of analytic automorphisms of a Riemann domain of values $D(Z)$ in solving a very general class of functional equations, including equations studied by Poincaré, Schroeder, Picard, Abel, and Böttcher, on permutability of rational fractions, linearizing of rational substitutions, etc. [see Julia, C. R. Acad. Sci. Paris **174**, 517–519, 653–655 (1922)]. *M. Morse.*

Landkof, N. S. On a harmonic invariant and the behavior of some bounded analytic functions near the boundary of a region. Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) **19**, 161–166 (1948). (Russian)

Let L_α denote the group of non-Euclidean rigid motions of the unit circle $|z| < 1$ into itself for which the point α, $(|\alpha| = 1)$, is the unique fix point. Let $u(z, \alpha)$ be a harmonic function in $|z| < 1$ which is invariant under all transformations of the group L_α. It is easily shown that then $u(z, \alpha) = AP(z, \alpha) + B$, where A and B are constants, while $P(z, \alpha)$ is the Poisson kernel:

$$P(z, \alpha) = \frac{1 - |z|^2}{|\alpha - z|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2}, \quad z = re^{i\varphi}, \; \alpha = e^{i\psi}.$$

If the further restriction is imposed on $u(z, \alpha)$ that it shall vanish everywhere on $|z| = 1$, except at $z = \alpha$, then it becomes a constant multiple of the Poisson kernel $P(z, \alpha)$. The author denotes any such function by $m(z, \alpha)$. If $f(z)$ is a regular analytic function in $|z| < 1$ for which $|f(z)| \leq 1$, then the map defined by $f(z)$ is said to have a fix point at α $(|\alpha| = 1)$ if there exists a sequence of points $\{z_n\}$, $|z_n| < 1$,

$z_n \to \alpha$, for which $f(z_n) \to \alpha$. It is shown that if α is the only fix point, then $m[f(z), \alpha] \geq m(z, \alpha)$, where the same constant multiplier is taken for both functions. Unless $f(z)$ defines a non-Euclidean rigid motion, the sign of equality cannot take place at an interior point. This result includes a part of the Julia-Wolff-Carathéodory theorem [see, e.g., Carathéodory, Conformal Representation, Cambridge University Press, 1932, p. 53]. By means of conformal mapping the theorem is extended to simply connected regions, bounded by closed Jordan curves. *W. Seidel* (Rochester, N. Y.).

Ahlfors, Lars, and Beurling, Arne. Conformal invariants and function-theoretic null-sets. Acta Math. **83**, 101–129 (1950).

Conformal invariants defined by extremal considerations are treated. One considers a class $\mathfrak{F}(\Omega)$ of functions analytic in a region Ω and introduces for $z_0 \varepsilon \Omega$ the quantity $M_{\mathfrak{F}} = M_{\mathfrak{F}}(z_0, \Omega) = \sup_{f \varepsilon \mathfrak{F}(\Omega)} |f'(z_0)|$. The following significant cases of $\mathfrak{F}(\Omega)$ are studied: $\mathfrak{B}(\Omega)$, the set of functions analytic and of modulus at most one in Ω; $\mathfrak{D}(\Omega)$, the set of functions analytic in Ω whose Riemannian images of Ω are of area $\leq \pi$; $\mathfrak{E}(\Omega)$, the set of functions analytic in Ω with the property that $[f(z) - f(z_0)]^{-1}$ omits a set of area $\geq \pi$. Further, $\mathfrak{SB}$, $\mathfrak{SD}$, and $\mathfrak{SE}$ denote the subclasses of $\mathfrak{B}$, $\mathfrak{D}$, $\mathfrak{E}$, respectively, consisting of the univalent functions of the corresponding classes (with constant functions admitted by convention as univalent). The following results are proved: $M_{\mathfrak{B}} = M_{\mathfrak{E}}$; $M_{\mathfrak{D}} = M_{\mathfrak{SD}}$, $M_{\mathfrak{SB}} = M_{\mathfrak{SE}}$; $M_{\mathfrak{SB}} \leq M_{\mathfrak{D}} \leq M_{\mathfrak{B}}$. If a closed set E and a point z_0 in the complement of E are given, E is termed a null set of class $N_{\mathfrak{F}}$ provided that $M_{\mathfrak{F}}(z_0, \Omega) = 0$, where Ω is the component of the complement of E which contains z_0. The inclusion relations $N_{\mathfrak{SB}} \supset N_{\mathfrak{D}} \supset N_{\mathfrak{B}}$ which follow from the above inequalities are shown to be proper by examples. The proof of the first equality is based on a specifically given mapping of a normalized subset of $\mathfrak{E}(\Omega)$ into $\mathfrak{B}(\Omega)$. The second equality is related to work of Grunsky [Schr. Math. Sem. u. Inst. Angew. Math. Univ. Berlin **1**, 95–140 (1932)] and Schiffer [Duke Math. J. **10**, 209–216 (1943); these Rev. **4**, 271] and the proof is based on methods involving the Dirichlet integral. The third equality is treated with the aid of the authors' theory of extremal length [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 341–351, Gjellerups, Copenhagen, 1947; these Rev. **9**, 23]. The paper settles a number of questions proposed by Sario [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 50 (1948); these Rev. **10**, 365] concerning null sets. *M. Heins.*

Schiffer, M., and Spencer, D. C. The coefficient problem for multiply-connected domains. Ann. of Math. (2) **52**, 362–402 (1950).

Known facts about a finite Riemann surface $\mathfrak{M}$ and its double $\mathfrak{F}$ are first reviewed, including the notion of a differential dZ^ν of dimension ν on $\mathfrak{F}$, and the method of Teichmüller [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1939, no. 22 (1940); these Rev. **2**, 187] of imbedding the divisor group of $\mathfrak{M}$ in the divisor group of $\mathfrak{F}$. The Green's function $g(p, q)$ of $\mathfrak{M}$ is set up with source at q, sink at $\bar{q}$ of $\mathfrak{F}$ from which certain functions and quadratic differentials are constructed and studied [see Schiffer, Duke Math. J. **13**, 529–540 (1946); these Rev. **8**, 371]. A variational formula is found for functions which are regular in a multiply-connected domain whose values are schlicht on another fixed multiply-connected domain. A generalization of Julia's variation formula [Ann. Sci. École Norm. Sup. **39**, 1–28 (1922)] is also obtained.

Let $\mathfrak{M}$ be a subdomain of a Riemann domain $\mathfrak{M}_0$ of genus zero with $m' > 2$ boundary curves. Let $\mathfrak{M}_0$ be mapped conformally on a fixed bounded multiply-connected domain $\mathfrak{R}$ of the plane with m' analytic boundary curves, then $f(z)$ is called an $\mathfrak{R}$-function if it is single-valued and analytic in $\mathfrak{M}$ and maps $\mathfrak{M}$ onto a schlicht subdomain of $\mathfrak{R}$. Let $\mathfrak{M}$ contain $z = 0$ and $\mathfrak{R}$ contain $w = 0$ with $f(z) = \sum_{n=1}^{\infty} b_n z^n$. For varying $\mathfrak{R}$-functions the point $(b_1, \cdots, b_n)$ fills out a Euclidean space V_n of $2n$ real dimensions, in general not connected, but bounded and compact. Let $F(b_1, \bar{b}_1, b_2, \bar{b}_2, \cdots, b_n, \bar{b}_n)$ be a continuous real function with continuous partial derivatives in an open set containing V_n and with nonvanishing gradient. Let $f(z)$ be an $\mathfrak{R}$-function maximizing F. Then f satisfies a relation between two quadratic differentials of $\mathfrak{R}$ and $\mathfrak{M}$, respectively. Moreover, $w = f(z)$ maps $\mathfrak{M}$ onto $\mathfrak{R}$ minus analytic slits. The characteristic curves lying on the boundary of V_n are determined. They satisfy a generalized Löwner differential equation. Results obtained earlier [see Schaeffer, Schiffer, and Spencer, Duke Math. J. 16, 493–527 (1949); these Rev. 11, 91] for functions $f(z)$ regular and schlicht in the interior of the unit circle are shown to fit in as a special case. The coefficient problem for functions mapping a multiply-connected domain onto a schlicht domain of the sphere is investigated. *M. S. Robertson.*

Kobori, Akira. An evaluation in the theory of multivalent functions. Proc. Japan Acad. 22, nos. 1–4, 75–77 (1946).
An announcement of results proved elsewhere [Jap. J. Math. 19, 275–285 (1948); these Rev. 11, 340].
A. W. Goodman (Lexington, Ky.).

Guseĭnov, A. I. On a nonlinear boundary problem of the theory of analytic functions. Mat. Sbornik N.S. 26(68), 237–246 (1950). (Russian)
The author considers the nonlinear problem for the unit circle (he terms it "nonlinear Hilbert problem"), with the boundary condition $l(\varphi)p + m(\varphi)q + \lambda\Phi(\varphi, p, q) = 0$, where $f(z) = p + iq$ is to be analytic interior to the unit circle, l, m, Φ are functions of Hölder type assigned on the circumference of the unit circle, and λ is a parameter. The solution is achieved with the aid of a singular integral equation and by making use of fixed point theorems; then, under suitable restrictions on Φ, a method of successive approximations yields the existence and uniqueness of the solution.
W. J. Trjitsinsky (Urbana, Ill.).

Gahov, F. D. On the Riemann boundary-value problem for systems of n pairs of functions with discontinuous coefficients. Doklady Akad. Nauk SSSR (N.S.) 73, 261–264 (1950). (Russian)
The problem treated is that of determining two n-tuples of functions
$$\varphi^+(z) = \{\varphi_1^+(z), \cdots, \varphi_n^+(z)\}, \quad \varphi^-(z) = \{\varphi_1^-(z), \cdots, \varphi_n^-(z)\},$$
which are regular analytic in the interior D_- and the exterior D_+, respectively, of a sufficiently smooth Jordan curve and whose boundary values $\varphi^+(t), \varphi^-(t)$ on L satisfy the conditions $\varphi^+(t) = C(t)\varphi^-(t)$. The elements of the nonsingular matrix $C(t)$ satisfy some Hölder (or some weaker "Magnaradze") condition except at a finite number of points $t_1, \cdots, t_m$, where they have discontinuities of the first kind. This problem has previously been solved by Hilbert [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1905, 307–338], Plemelj [Monatsh. Math. Phys. 19, 211–246 (1908)], Vekua [Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoĭ SSR] 5, 1–10 (1944); these Rev. 7, 203], and

Magnaradze [Doklady Akad. Nauk SSSR (N.S.) 64, 17–20 (1949); these Rev. 10, 439]. The author's method differs from Vekua's in that he uses functions of the matrices $C^{-1}(t_k+0)C(t_k-0)$ without reference to a canonical form.
M. Golomb (Lafayette, Ind.).

Conforto, Fabio. Sulla totalità delle relazioni generalizzate di Hurwitz di una matrice quasi abeliana. Ann. Mat. Pura Appl. (4) 28, 299–315 (1949).
Sia ω la matrice (con π righe e $\pi' < 2\pi$ colonne) dei periodi di un corpo di funzioni quasi-abeliane di π variabili $u_1, \cdots, u_r$. L'autore considera le cosiddette relazioni generalizzate di Hurwitz relative ad ω, del tipo: $\Lambda\omega = \omega I$, essendo Λ, I matrici ad elementi rispett. complessi e interi. Rimandando a successivi lavori le considerazioni funzionali collegate con le predette relazioni, l'autore ne approfondisce lo studio puramente aritmetico, mostrando le divergenze che si presentano rispetto al caso delle funzioni abeliane. In particolare vengono determinate le condizioni necessarie e sufficienti affinché la I individui Λ o sia arbitraria. *E. Martinelli* (Genova).

Fueter, Rud. Ueber Abelsche Funktionen von zwei Komplexen Variablen. Ann. Mat. Pura Appl. (4) 28, 211–215 (1949).
Siano $f_1(z_1, z_2)$, $f_2(z_1, z_2)$ funzioni analitiche delle variabili complesse coordinarie $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$. Partendo dall'osservazione che $f_1 + jf_2$ rientra come caso particolare nella classe delle cosidette funzioni analitiche (o regolari) della variabile quaternione $z = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3$, l'autore é tratto a subordinare la teoria delle funzioni abeliane di 2 variabili complesse a quella delle funzioni analitiche di quaternione quattro volte periodiche e meromorfe. Per queste ultime funzioni é assegnata una rappresentazione generale fondata sulla considerazione di una speciale funzione $\zeta(z)$, analoga alla funzione ζ di Weierstrass nella teoria delle funzioni ellittiche. *E. Martinelli* (Genova).

Cartan, Henri. Idéaux et modules de fonctions analytiques de variables complexes. Bull. Soc. Math. France 78, 29–64 (1950).
Die Arbeit schliesst die Untersuchungen des Verf. [Ann. Sci. École Norm. Sup. (3) 61, 149–197 (1944); diese Rev. 7, 290] und von K. Oka [Bull. Sci. Math. France 78, 1–27 (1950); diese Rev. 12, 18] ab. Das Ziel der Arbeit ist das Studium der Eigenschaften von Moduln und Idealen analytischer Funktionen im Grossen. Dazu werden zunächst die lokalen Zusammenhänge studiert. Verf. spricht von einem Bündel (faisceau) von Moduln, wenn bei vorgegebenen ganzen Zahlen n und q jeder nicht leeren Untermenge A im Raume von n komplexen Veränderlichen ein q-dimensionaler Modul $\mathfrak{F}_A$ so zugeordnet ist, dass, wenn $A \subset B$, der durch $\mathfrak{F}_B$ in A erzeugte Modul in $\mathfrak{F}_A$ enthalten ist. Dann wird zunächst bewiesen: (1) Das Bündel der koeffizienten linearer Gleichungen von in A regulären Funktionen f_i, $1 \leq i \leq p$, ist in A zusammenhängend; (2) das Bündel der Ideale einer analytischen Mannigfaltigkeit in einem offenen Bereiche A ist in A zusammenhängend.
Im nächsten Abschnitt findet dann der Übergang vom Kleinen ins Grosse mit Hilfe Cauchyscher Integrale statt. Zunächst werden als Gebiete nur abgeschlossene Pflaster (domaines pavés), sodann abgeschlossene Polyedergebiete und schliesslich beliebige offene Regularitätsgebiete zugelassen. Die Hauptsätze sind die folgenden. (1) $f_1, \cdots, f_p$ seien im Regularitätsgebiet D regulär. Dann erzeugt der Modul der in D regulären Funktionen $c_1, \cdots, c_p$ mit $\sum c_i f_i = 0$

in jedem Punkte P von D den Modul aller linearen Relationen $\sum c_i f_i = 0$ dieses Punktes P. (2) Die Gesamtheit der in P regulären Funktionen, die auf einer analytischen Mannigfaltigkeit V in D verschwinden, hat in D keine weiteren Nullstellen. (3) Die Funktion f sei in einer Nachbarschaft V einer analytischen Mannigfaltigkeit in D regulär. Dann gibt es eine in D reguläre Funktion, die auf V gleich f ist.
H. Behnke (Münster).

Hermes, Hans, und Peschl, Ernst. Über analytische Automorphismen des R_{3n}. Math. Ann. **122**, 66–70 (1950).

In the space E_{2k} of k complex variables $z_1, \cdots, z_k$, $k \geqq 2$, there are complex-holomorphic isomorphisms which are non-affine. The authors call a sequence of points $\{P_n\}$ in E_{2k} "plane-like" (planierbar) if by such an isomorphism they can be carried into a sequence which is located in an E_2 as given by $z_j = a_j t + b_j$, t complex. They prove that any sequence without a finite limit point is plane-like and they also prove that there is an entire function $E(z_1, \cdots, z_k)$ for which $E(P_n) = a_n$, where the numbers $\{a_n\}$ are prescribed arbitrarily.
S. Bochner (Princeton, N. J.).

Sakai, Eiichi. Note on pseudo-analytic functions. Proc. Japan Acad. **25**, no. 9, 12–17 (1949).

Known theorems concerning univalent analytic functions and cluster sets of meromorphic functions are extended to the more general corresponding classes of pseudo-analytic functions [Ahlfors, Acta Math. **65**, 157–194 (1935); Kakutani, Jap. J. Math. **13**, 375–392, 393–404 (1937)].
M. Heins (Providence, R. I.).

Šabat, B. V. Cauchy's theorem and formula for quasi-conformal mappings of linear classes. Doklady Akad. Nauk SSSR (N.S.) **69**, 305–308 (1949). (Russian)

The author derives a Cauchy theorem and a Cauchy integral formula for a pair of functions $u(x, y)$, $v(x, y)$ connected by the equations

$$(1) \qquad au_x + bu_y - v_y = 0, \quad du_x + cu_y + v_x = 0,$$

where a, b, c, d are continuously differentiable functions of (x, y), and $4ac - (b+d)^2 > 0$ (ellipticity condition). Denote the right hand sides of (1) by $L(u, v)$, $M(u, v)$ and let L^*, M^* denote the operators obtained from L, M by interchanging b and d, and L_1, M_1 the operators obtained by replacing a, b, c, d by $-a/B$, $-b/B$, $-c/B$, $-d/B$, $B = ac - bd$. Let u, v be continuously differentiable solutions of (1), let X, Y and X_1, Y_1 be twice continuously differentiable solutions of the systems $L^*(X, Y) = 0$, $M^*(X, Y) = 0$, and $L_1(X_1, Y_1) = 0$, $M_1(X_1, Y_1) = 0$, respectively, and set $Z = X + iY_1$, $Z^* = X_1 + iY$. Then $\int_\Gamma u dZ^* + iv dZ = 0$, where the curve Γ satisfies the conditions of Cauchy's theorem. The generalized Cauchy formula reads

$$u(z_0) + iv(z_0) = (2\pi i)^{-1} \int_\Gamma u(z) d_z l^*(z, z_0) + iv(z) d_z l(z, z_0).$$

Here $l^* = \Gamma_1 + iH$, $l = \Gamma + iH_1$, Γ and H are connected by the equations (2) $L^*(\Gamma, H) = 0$, $M^*(\Gamma, H) = 0$, $\Gamma(z, z_0)$ is an appropriately normalized fundamental solution of the second order equation obtained from (2) by elimination, and Γ_1, H_1 are obtained in the same way from the system $L_1(\Gamma_1, H_1) = 0$, $M_1(\Gamma_1, H_1) = 0$. The author's methods and results generalize those due to Položil for the case $a = c$, $b = d = 0$ [same Doklady (N.S.) **58**, 1275–1278 (1947); Mat. Sbornik N.S. **24(66)**, 375–384 (1949); these Rev. **9**, 507; **11**, 171]. For this

case a different approach to Cauchy's formula, independent of the theory of the fundamental solution, has been announced by the reviewer [see the following review].
L. Bers (Princeton, N. J.).

Bers, Lipman. Partial differential equations and generalized analytic functions. Proc. Nat. Acad. Sci. U. S. A. **36**, 130–136 (1950).

This note [an abstract of results to be published in full later] presents the fundamentals of a theory of complex-valued functions $f = u(x, y) + iv(x, y)$ whose real and imaginary parts are connected by the equations (1) $u_x = \sigma(x, y) v_y$, $u_y = -\sigma(x, y) v_x$, a theory which parallels closely that of analytic functions. Set $x + iy = z$ and write functions of x and y as functions of z, without implying an analytic dependence on z. The function $\sigma(z) = \sigma(x, y)$ is supposed to be positive in the whole complex plane (Riemann sphere) and σ_x and σ_y are supposed to exist and satisfy a Hölder condition. A complex-valued function $f(z)$ is said to be pseudo-analytic with respect to σ in an open set D if and only if the partial derivatives of u and v are continuous in D and satisfy (1) throughout D. With each complex-valued function $g(z) = \varphi(x, y) + i\psi(x, y)$, let the "reduced" function $^{\sigma(z)}g(z) = \sigma(x, y)^{-1}\varphi(x, y) + i\sigma(x, y)^{\frac12}\psi(x, y)$ be associated. Using this notation, pseudo-analyticity may be characterized as a differentiability requirement: $f(z)$ is pseudo-analytic in an open set D if and only if for every z_0 in D the function $^{\sigma(z)}[f(z) - f(z_0)]$ (considered as a function of the complex variable z) is differentiable at z_0. Various results, analogous to results in ordinary complex variable theory, are announced. For each rational number r, and complex numbers a and ζ, there exists a uniquely determined pseudo-analytic function $Z^{(r)}(a, \zeta; z)$, called a formal power with exponent r, coefficient a, and center at ζ, which "corresponds" to the analytic function $a(z-\zeta)^r$. With the aid of these formal powers, differential quotients of any order are defined for any pseudo-analytic function $f(z)$, even if f, considered as a function of x and y, does not possess partial derivatives of order higher than the second. Theorem: If $f(z)$ is pseudo-analytic and all its "pseudo-analytic differential quotients vanish at a point, then $f(z) \equiv 0$. Theorem: A formal polynomial of degree n, $\sum_{r=0}^{n} Z^{(r)}(a_r, \zeta; z)$, $a_n \neq 0$, is uniquely determined by its center, its leading coefficient, and its (exactly) n zeros. Formal power series expansions of pseudo-analytic functions are then considered. The following analogue of Cauchy's formula is given: $f(z) = (2\pi)^{-1} \int_C Z^{-1}\{[^{\sigma(\zeta)}f(\zeta)]id\zeta, \zeta; z\}$. A consequence of this formula is the Laurent expansion $f(z) = \sum_{r=-\infty}^{+\infty} Z^{(r)}(a_r, \zeta; z)$ for single-valued functions $f(z)$ which are pseudo-analytic in a ring $0 < |\zeta - z| < R$. There is a similar expansion $f(z) = \sum_{r=-\infty}^{+\infty} Z^{(r/k)}(a_r, \zeta; z)$ for a pseudo-analytic $f(z)$ which is k-valued on the ring $0 < |\zeta - z| < R$. The note concludes with a theorem concerning the uniformization of pseudo-analytic functions.
J. B. Diaz.

Fourier Series and Generalizations, Integral Transforms

Iliev, Lübomir. Über trigonometrische Polynome mit monotoner Koeffizientenfolge. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 87–102 (1942). (Bulgarian. German summary)

Izumi, Shin-ichi. Notes on Fourier analysis. XXI. On the degree of approximation of the partial sums of a Fourier series. J. London Math. Soc. 25, 240–242 (1950).

If $0<\alpha\leq1$ and $\alpha p>1$, then the partial sum $S_n(x)$ of the Fourier series of $f(x)\epsilon\mathrm{Lip}\,(\alpha, p)$ satisfy

(*) $f(x)-S_n(x)=O(n^{1/p-\alpha})$

almost everywhere (uniformly in x), even though (*) is not satisfied in general by functions in $\mathrm{Lip}\,(\alpha-1/p)$. If, in addition, $1<p\leq2$, and $\{C_n\}$ are the Fourier coefficients of $f(x)$, it is shown that $\sum_{|n|>M}|C_n|=O(M^{1/p-\alpha})$.

P. Civin (Eugene, Ore.).

Tsuchikura, Tamotsu, and Yano, Shigeki. On the absolute convergence of trigonometrical series. Proc. Amer. Math. Soc. 1, 517–521 (1950).

Let S be the set of points in the interval $[0, \pi]$ at which the series $\sum_{n=1}^{\infty}\rho_n \cos (nx-\alpha_n)$, $\rho_n\geq0$, converges absolutely and $\mathcal{S}$ the corresponding set for the conjugate series $\sum_{n=1}^{\infty}\rho_n \sin (nx-\alpha_n)$. Then $x_0\epsilon S$, $x_1\epsilon\mathcal{S}$ implies $S=[0, \pi]$ if and only if $x_0-x_1=p\pi/q$, where p/q is an irreducible rational with q odd. If $\sum_{n=1}^{\infty}\rho_n=\infty$, and either $\rho_{n+b}/\rho_n=O(1)$ for $b>0$ or $\sum_{j=1}^{n}1/\rho_j=O(n^2)$, then $S\neq0$ implies $\mathcal{S}=0$ (and S is a single point), while $\sum_{n=1}^{\infty}\rho_n=\infty$ and $\rho_n=O(1/n)$ implies S is of α-capacity zero for $0<\alpha<1$. In both conclusions of theorem 3 of the paper, the term "converge" should read "converge absolutely."

P. Civin (Eugene, Ore.).

Sunouchi, Genichirô. Notes on Fourier analysis. XXV. Quasi-Tauberian theorem. Tôhoku Math. J. (2) 1, 167–185 (1950).

Hardy and Littlewood proved that a necessary and sufficient condition for the Fourier series of a function $f(t)$ to be summable by some Cesàro mean for $t=x$ is that $f(x+t)+f(x-t)$ should tend to a limit in the Cesàro sense for some order as $t\to0$; they also proved a similar result for the allied (conjugate) series of a Fourier series [Math. Z. 19, 67–96 (1923); Proc. London Math. Soc. (2) 24, 211–246 (1925); Proc. Cambridge Philos. Soc. 23, 681–684 (1927)]. Many refinements, analogues, and generalizations have been given by later writers. The author refers to some of these in his bibliography. In particular, Wiener [Proc. London Math. Soc. (2) 30, 1–8 (1929); Ann. of Math. (2) 33, 1–100 (1932)] deduced Hardy and Littlewood's theorem from a general quasi-Tauberian theorem. Here the author gives twenty theorems in this field, some known and some new, involving summability and absolute summability of Fourier series, allied series, derived Fourier series and allied series, Fourier integrals, and generalized jumps. His main weapons are Wiener's quasi-Tauberian theorem concerning limits of means and an analogue of his own concerning absolute limits (bounded variation) of means. In the statement of the latter (theorem 1') "dy" should be "d_y."

L. S. Bosanquet.

Rényi, Alfréd. On the summability of Cauchy-Fourier series. Publ. Math. Debrecen 1, 162–164 (1950).

Let $tf(t)\epsilon L(0, \pi)$, where $f(t)$ is odd and of period 2π, and let $b_n=(2/\pi)\int_0^{\pi}f(t) \sin nt\,dt$. Titchmarsh [Proc. London Math. Soc. (2) 23, xli–xliii (1924)] remarked that $b_n=o(n)$ and that, for $t\not\equiv0$ (mod 2π), the principal value Fourier series (*) $\sum b_n \sin nt$ is summable $(C, 1)$ if and only if $f(t)$ satisfies a local $(C, 1)$ summability condition. Here the author (a) shows that, for $t\neq\pi$, (*) is summable $(C, 1)$ if and only if the Fourier series of $f(t) \sin t$ is summable $(C, 1)$ and (b) observes that (*) is the derived Fourier series of an

integral of $f(t)$ for $t\not\equiv2\pi$, so that Titchmarsh's statement follows from W. H. Young's theory of restricted Fourier series [ibid. (2) 17, 195–236 (1918)]. The author's formula (7) seems to require some modification.

L. S. Bosanquet (London).

Obrechkoff, Nikola. Sur les moyennes arithmétiques des séries trigonométriques. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 38, 103–192 (1942). (Bulgarian. French summary)

Cheng, Min-Teh. Uniqueness of multiple trigonometric series. Ann. of Math. (2) 52, 403–416 (1950).

Le résultat essentiel est le suivant. Soit

$$\sum a_{m_1,\ldots,m_k} \exp (i(m_1x_1+\cdots+m_kx_k))$$

une série trigonométrique à k variables. Si la suite des sommes sphériques $S(R)=\sum_{m_1^2+\cdots+m_k^2\leq R}$ converge par moyennes arithmétiques vers 0, et si $a_{m_1,\ldots,m_k}=O(r^{-k+1-\epsilon})$, $r=m_1^2+\cdots+m_k^2$, alors tous les coefficients $a_{m_1,\ldots,m_k}$ sont nuls. Plus généralement, si les sommes sphériques convergent par moyennes vers une fonction $f(x_1, \cdots, x_k)$, on peut, dans certaines conditions, affirmer que la série est la série de Fourier de f. Les démonstrations utilisent une résolution de l'équation de Laplace $\Delta U=V$, où V est assez régulière, Δ étant un laplacien généralisé.

L. Schwartz (Nancy).

Kozlov, V. Ya. On complete systems of orthogonal functions. Mat. Sbornik N.S. 26(68), 351–364 (1950). (Russian)

Let $\{\varphi_n(x)\}$ be orthonormal and complete in $L^2(0,1)$. The author investigates the completeness of subsystems on subsets of measure less than 1, and gives applications to trigonometric series. (1) If $\{n_k\}$ is a sequence of integers composed of products of powers of the first q primes, then deleting all the functions with indices n_k from the sequence $\{e^{2\pi inx}\}$ leaves a system which is complete on any set of measure less than 1. (2) There is a countable set of uniqueness for lacunary series $\sum a_n \sin (2\pi\cdot2^nx)$. (3) Let $\{\varphi_n(x)\}$ be called essentially linearly independent if every finite subsystem is linearly independent on every set of positive measure. If from such a system which is orthonormal and complete we delete a finite number of functions, the reduced set is complete on any set of measure less than 1. (4) Under the hypotheses of (3), a given function may be changed on a given set of positive measure so that it becomes orthogonal to the first N functions of the system. (5) There is a (nonzero) trigonometric series with a sequence of partial sums converging everywhere to zero. (6) There is a "universal" trigonometric series such that to any odd measurable function $f(x)$ on $(0, 1)$ there is a subsequence of partial sums of the series converging to $f(x)$ almost everywhere, and uniformly in any $(\delta, \frac{1}{2}-\delta)$ if $f(x)$ is continuous. (7) There is a trigonometric series with a sequence of partial sums diverging to ∞ in $(0, \frac{1}{2})$.

R. P. Boas, Jr. (Evanston, Ill.).

✦**Cinquini, Silvio.** Funzioni quasiperiodiche. Scuola Normale Superiore, Pisa. Quaderni Matematici, no. 4. Litografia Tacchi, Pisa, 1950. 132+7pp.

This is a very lucid introduction to the "classical" theory of almost periodic functions on the straight line with emphasis on Stepanoff's generalization of the Bohr functions. The uniqueness theorem is proved more or less by the method of de La Vallée Poussin, and the approximation theory is the reviewer's. The nonstandard theorems given include among others a convergence statement on $\int\psi(|\sigma_m|)dx$,

where $\psi(t)$ is a convex function and $\{\sigma_m\}$ are approximating exponential polynomials, and conditions for the validity of

$$\lim_{n\to\infty}\lim_{-\infty<\alpha<\infty}\sup \varphi\left[\int_\alpha^{\alpha+1}\psi(|f(x)-\sigma_m(x)|)dx\right]=0,$$

where $t=\varphi(u)$ is the inverse function to $u=\psi(t)$.

S. Bochner (Princeton, N. J.).

Kawata, Tatsuo, and Udagawa, Masatomo. Some gap theorems. Kōdai Math. Sem. Rep., no. 5–6, 19–22 (1949).

The author proves the following two theorems. (I) If $\sum_1^\infty(a_k\cos\lambda_kx+b_k\sin\lambda_kx)$, $\lambda_{k+1}/\lambda_k>\lambda>1$, is a Fourier series of a bounded S^2 almost periodic function, then

$$\sum_1^\infty(|a_k|+|b_k|)<\infty.$$

(II) If $\varphi(x+2\pi)=\varphi(x)$, $|\varphi(x')-\varphi(x'')|<M|x'-x''|^a$, $\int_0^{2\pi}\varphi(x)dx=0$, $\sum_1^\infty c_k^2<\infty$, $\lambda_{k+1}/\lambda_k>\lambda>1$, then $\sum_1^\infty c_k\varphi(\lambda_kx)$ converges almost everywhere. The first theorem is an extension to the nonharmonic case of a well-known theorem of Szidon and the second generalizes a result of the reviewer [Ann. of Math. (2) **44**, 411–415 (1943); these Rev. **5**, 4], who proved (II) under the additional assumption that the λ_k's are integers. The proofs utilize Hartman's improvement [Duke Math. J. **9**, 404–405 (1942); these Rev. **4**, 39] of a method of the reviewer [ibid. **8**, 541–545 (1941); these Rev. **3**, 107].

M. Kac (Ithaca, N. Y.).

Guinand, A. P. A class of Fourier kernels. Quart. J. Math., Oxford Ser. (2) **1**, 191–193 (1950).

The author shows that a function $K(x)$ is a Fourier kernel if its Mellin transform is $2^k(2\pi)^{-1}\Gamma(s)\prod\sin\frac{1}{2}(s+n)\pi/k$, where the product is over a set of k integers n such that $0\leq n\leq 2k-1$ and n and $2k-1-n$ are never both members of the set. Such a $K(x)$ is of the form

$$\sum_{r=0}^k A_r e^{-x\cos(r\pi/2k)}\cos\{x\sin\frac{1}{2}r\pi/k+\frac{1}{2}m_r\pi/k\},$$

where m_r are integers. As specific examples the author gives the sine and cosine kernels $(k=1)$,

$$K(x)=\pi^{-\frac{1}{2}}(e^{-x}-\cos x+\sin x)$$

$(k=2)$ and the six new cases arising for $k=3$.

R. P. Boas, Jr. (Evanston, Ill.).

Widder, D. V. An inversion of the Lambert transform. Math. Mag. **23**, 171–182 (1950).

The Lambert transform $F(x)$ of $a(t)$ is defined by the equation $F(x)=\int_0^\infty t[e^{xt}-1]^{-1}a(t)dt$, where $a(t)$ is assumed to be Lebesgue integrable for $0\leq t\leq R$, for every positive R. The present paper contains a study of the convergence and inversion of the Lambert transform. It is shown that if the integral defining $F(x)$ converges conditionally for any value of $x>0$, then it converges for all larger values of x, and that if it converges for $x=0$, then it converges for all x. If the additional restrictions

$$\int_0^\infty|a(t)|dt<\infty,\quad \limsup_{t\to0+}\log|a(t)|/\log(1/t)<1,$$

are imposed on $a(t)$ then, at any point of continuity t_0 of $a(t)$, the Lambert transform is inverted by the formula

$$t_0a(t_0)=\lim_{k\to\infty}\frac{(-1)^k}{k!}\left[\frac{k}{t_0}\right]^{k+1}\sum_{n=1}^\infty\mu(n)n^kF^{(k)}\left[\frac{nk}{t_0}\right],$$

where $\mu(n)$ is the Möbius function. An explanation for the form of this inversion formula is obtained when the Lambert transform is, by a change of variable, turned into a convolution transform with kernel $(e^{-x}-1)^{-1}$. Since $\int_{-\infty}^\infty(e^{-t}-1)e^{-st}=\Gamma(s)\zeta(s)$, $[\Gamma(D)\zeta(D)]^{-1}$ is a symbolic inversion operator. Suitably interpreted, this leads to the formula given above.

I. I. Hirschman, Jr.

Good, I. J. Bounded integral transforms. II. Quart. J. Math., Oxford Ser. (2) **1**, 185–190 (1950).

This is a continuation of a paper by the author and Reuter [same Quart., Oxford Ser. (1) **19**, 224–234 (1948); these Rev. **10**, 451] in which it was proved that every linear bounded transformation on $L_2(0,\infty)$ to itself is of the form $T[f]=T_\phi[f]=(d/dx)\int_0^\infty\phi(x,t)f(t)dt$, with a so-called standard kernel ϕ satisfying certain simple specified conditions. In the present note the author determines the standard kernel of T_2T_1 to be $(d/dx)\int_0^\infty\psi(x,t)\phi^*(x,t)dt$, where ψ and ϕ^* are the kernels of T_2 and T_1^*, respectively. He also shows that $\phi(x,t)$ is a standard kernel if and only if there exists a standard kernel Φ such that

$$\int_0^\infty\phi(x,t)\phi(y,t)dt=\int_0^\infty\Phi(y,t)dt.$$

If this is the case, $\|T_\phi\|^2=\|T_\Phi\|$ and T_Φ is self-adjoint. The rest of the paper contains remarks by Kober giving alternate or slightly more general formulations. The author also presents an iterative process for inverting bounded transformations similar to methods used for finding the inverse of a matrix [see Hotelling, Ann. Math. Statistics **14**, 1–34 (1943); these Rev. **4**, 202].

E. Hille.

Berti, Giuliana. Qualche proprietà di alcuni operatori introdotti dal Volterra. Boll. Un. Mat. Ital. (3) **4**, 279–281 (1949).

The operators $C_r=[c_r,\gamma_r]$ are considered, which applied to a function $f(t)$ continuous for $t\geq0$ give the result: $C_rf(t)=c_rf(t)+\int_0^t\gamma_r(t,\tau)f(\tau)d\tau$, c_r being a number and γ_r a function, continuous for $t\geq0$, $\tau\geq0$. The sum and the product of these operators are defined by $C_r+C_s=[c_r+c_s,\gamma_r+\gamma_s]$, $C_rC_s=[c_rc_s,c_r\gamma_s+c_s\gamma_r+\dot\gamma_r.\dot\gamma_s]$, where $\dot\gamma_r.\dot\gamma_s$ is the composition product, according to Volterra, of γ_r and γ_s. For these operations the associative and distributive properties are demonstrated and they follow as a consequence of the analogous properties for the composition product. The commutative property is then proved only under the condition that the γ_r and γ_s are functions of the difference $t-\tau$.

C. Miranda (Naples).

Polynomials, Polynomial Approximations

Kuipers, L. Note on the location of zeros of polynomials. Nederl. Akad. Wetensch., Proc. **53**, 482–486 = Indagationes Math. **12**, 134–138 (1950).

The author determines the region C in which are located all the zeros of the linear combination $h(z)=\lambda f(z)+g(z)$ of two given polynomials $f(z)=(z-a_1)\cdots(z-a_n)$ and $g(z)=(z-b_1)\cdots(z-b_n)$ when all the a_k are points of a circular region A and all the b_j are points of a circular region B. His first two theorems are the following. (I) If $A:|z-\alpha|\leq R$, $B:|z-\alpha|>\rho$, $\rho<R$; then $C:|z-\alpha|>(\rho-Rt)/(1+t)$, $|\lambda|^{1/n}\leq t<\rho/R$. (II) If $A:|z-\alpha|\leq\rho_1$, $B:|z-\beta|>\rho_2$, $\rho_2<\rho_1$, $|\alpha-\beta|<\min(\rho_1-\rho_2,\rho_3)$, then $C:|\lambda^{1/n}|\cdot|z-\alpha|+|z-\beta|>\rho_2-|\lambda|^{1/n}\rho_1$. In the third theorem

$\lambda = 1$, A: $|s-\alpha| \leq d$, B is a half-plane bounded by a line L but not containing any points of A, and C is the complement of the regions common to the interior of a parabola P with focus at an arbitrary point β not in A or B with directrix L, and to the interior of the hyperbola H with foci α and β and transverse axis d. The fourth is a similar theorem in which $|\lambda| < 1$ and $g(s) = (s-\alpha)^n$ and in which the hyperbola H is replaced by the circle C: $|s-\alpha| = |\lambda|^{1/n}|s-\beta|$. The final theorem concerns the location of the zeros of the derivative of the rational function $F(s) = f(s)/g(s)$ when $\Re(a_j) = \Re(b_j)$ for all j. The proofs of all five theorems are based upon the use of elementary inequalities. [Reviewer's note: Theorem I is a special case of theorem (17, 2b), p. 58, of M. Marden's The Geometry of the Zeros . . . [Mathematical Surveys, no. 3, Amer. Math. Soc., New York, 1949; these Rev. 11, 101], and theorem II does not furnish as good a result as this theorem. Theorems III and IV are somewhat related to theorems (17, 3a), (17, 3b), and (17, 3c) [loc. cit.].]   M. Marden (Milwaukee, Wis.).

Sz.-Nagy, Gyula. Apollonische Kurven. Publ. Math. Debrecen 1, 73–88 (1949).

The author studies various properties of the Apollonian curves $C_n(\alpha, \beta; T)$ defined by the equation $|F(s)| = T$, where $F(s) = f(s)/g(s)$ with $f(s) = (s-\alpha_1) \cdots (s-\alpha_n)$ and $g(s) = (s-\beta_1) \cdots (s-\beta_n)$ and $f(\beta_j)g(\alpha_k) \neq 0$ for all j and k. The singular points of C_n are at the zeros of $F'(s)$. Denoting by $H(\varphi)$ the set of zeros c_j of the polynomial $Q(s) = f(s) - Te^{i\varphi}g(s)$, the author proves that the center of gravity of $H(\varphi)$ is either independent of φ or lies on the curve $C_1(\zeta_\alpha, \zeta_\beta; T)$, where ζ_α and ζ_β are respectively the centers of gravity of the α_j and the β_k. Denoting by c_j' the members of the set $H(\varphi')$ and defining $\Omega = \sum_1^n \omega_k \pmod \pi$ with $\omega_k = \arg(s-c_k')/(s-c_k)$, he shows that, for every point s of C_n, $\Omega = \frac{1}{2}(\varphi' - \varphi) - \Psi \pmod \pi$, where Ψ is the angle subtended at $s=1$ by the vector drawn from point $Te^{i\varphi}$ to point $Te^{i\varphi'}$. Some topological properties of the curves C_n, as well as their relation to the circles $|(s-\alpha_k)/(s-\beta_k)| = t_k$ with $t_1 \cdots t_n = T$, are also discussed. The proofs are all fairly elementary.   M. Marden (Milwaukee, Wis.).

Bernšteĭn, S. N. On some new results in the theory of approximation of functions of a real variable. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 161–169 (1950). (Russian)

The author gives a connected summary of recent results, mainly due to himself, on approximation by polynomials and by entire functions of exponential type.
    R. P. Boas, Jr. (Evanston, Ill.).

Favard, J. Remarques sur l'approximation des fonctions continues. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 101–104 (1950).

The author discusses the derivatives of the Bernstein polynomials and deduces various facts about polynomial approximation; for example, a function on $(-1, 1)$ with positive pth derivative can be approximated by polynomials with the same property, using only the values $f(k/n)$. As an application he deduces the differentiability properties of functions subject to restrictions on their pth differences. Finally he shows that for any dense sequence of interpolation points $\{s_k^n\}$ there is a sequence of polynomials $\{B_k^n(x)\}$ such that $\sum_{k=0}^n f(s_k^n)B_k^n(x)$ approaches a continuous $f(x)$ uniformly and is convex if $f(x)$ is convex.
    R. P. Boas, Jr. (Evanston, Ill.).

Mergelyan, S. N. On best approximation in a complex domain. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 198–212 (1950). (Russian)

The author considers the best approximation $\rho(f, D)$ by polynomials of degree n to an $f(s)$ which is regular in the region D and continuous on the boundary. The following results are established. (1) If the boundary of D is a smooth curve with continuously turning tangent and $f^{(k)}(s)\varepsilon\text{Lip}\,\alpha$, $0 < \alpha \leq 1$, in $\bar{D}$, then $\rho_n(f, D) < Cn^{-k-\alpha+\epsilon}$, $\epsilon > 0$. (2) If $\omega(\delta)$ is the modulus of continuity of $f(s)$ in $\bar{D}$, then $\rho_n(f, D) < C\omega(n^{-1+\epsilon})$. (3) Let the boundary of D be the curve $s = s(s)$, s arc length, and let $\gamma(\delta)$ be the modulus of continuity of $s'(s)$. If

$$\int_s^s x^{-1}\gamma(x)dx > |\log|\log \epsilon|| \cdot |\log|\log|\log \epsilon|||,$$

then "in general" there is a $\varphi(s)$ of Lip α such that $\lim \sup \rho_n(\varphi, D)n^\alpha(\log n)^{-\alpha} = \infty$. (4) Results similar to (3) for a special form of $\gamma(\delta)$. (5) If $\int_s x^{-1}\gamma(x)dx$ converges and $\rho_n(f, D) < Cn^{-k-\alpha}$, $0 < \alpha < 1$, then $f^{(k)}(s)\varepsilon\text{Lip}\,\alpha$ in $\bar{D}$. If $\alpha = 1$, $f^{(k)}(s)\varepsilon\text{Lip}\,1$ if and only if $\sum_{n=1}^\infty n^k\rho_n(f, D) < \infty$. (6) A further theorem inferring properties of $f(s)$ from the magnitude of $\rho_n(f, D)$ under hypotheses too complicated to reproduce here. This theorem implies that $\rho_n(f, D)$ may tend to zero arbitrarily slowly and still imply that $f(s)$ is infinitely differentiable at a boundary point if D behaves suitably near that point.   R. P. Boas, Jr. (Evanston, Ill.).

Geronimus, Ya. L. Polynomials orthogonal on a circle and their applications. Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) 19, 35–120 (1948). (Russian)

With a nondecreasing function $\sigma(\theta)$ on $(0, 2\pi)$ one may associate its Fourier coefficients (moments) c_n, the set of polynomials $P_n(s) = s^n + \cdots$ orthogonal on the unit circle with weight function $d\sigma(\theta)$, the C(arathéodory)-function

$$F(s) = (2\pi c_0)^{-1}\int_0^{2\pi} (e^{i\theta}+s)(e^{i\theta}-s)^{-1}d\sigma(\theta)$$

(with positive real part in $|s| < 1$), and the S(chur)-function $f(s) = s^{-1}[F(s)-1]/[F(s)+1]$ (bounded by 1 in $|s| < 1$). This paper is a detailed and systematic survey in which the $P_n(s)$ are taken as fundamental and applied to the study of $\{c_n\}$, $F(s)$, and $f(s)$, as well as to the converse problem of deciding when a given sequence is a trigonometric moment sequence or when a given function is a C- or S-function. There are also applications to polynomials orthogonal on a real interval. The principal known results are reproduced, in many cases with new proofs, and new results are obtained. In particular, for the trigonometric moment problem there are analogues of several results which are well known for the ordinary moment problem, and these lead to new results on C- and S-functions. Connections with the theory of continued fractions are pointed out at appropriate places.

Additional notation:

$$P_k^*(s) = s^k P_k(1/s); \quad a_k = (-1)^k |c_{l-j+1}|_0^k/\Delta_k, \quad \Delta_k = |c_{l-j}|_0^k.$$

We quote a selection of the topics which the author discusses in addition to formal properties (some of which seem to be new) and standard material. The value region for $F(s)$ with a given set of initial moments $\{c_k\}_0^n$. Necessary and sufficient conditions for a function to be a C-function (in terms of its power series); necessary and sufficient conditions for a trigonometric series to be the Fourier series of the reciprocal of a positive trigonometric polynomial. Gen-

eralization of Julia's inequality for S-functions: if $|f(z)|<1$ and $\lim_{z\to z_s} f(z)=1$, $\lim [f(z)-1]/(z-z_s)=\beta_s$ for s values z_s on the unit circle, then

$$\frac{1-|f(z)|^2}{|1-f(z)|^2} \geq \Re \sum_1^s \frac{1}{\beta_s z_s} \frac{z_s+z}{z_s-z}.$$

Properties of the solution of the trigonometric moment problem with given parameters a_k: for example, if

$$\limsup \left\{ \prod_{k=0}^{n-1} (1-|a_k|^2) \right\}^{1/n} = 1$$

(and in particular if $a_k \to 0$) the set of points of increase of $\sigma(\theta)$ is dense in $(0, 2\pi)$; corresponding results for C- and S-functions. Asymptotic formulas for $P_n(z)$ when $\sum |a_n|^2 < \infty$ (six equivalent conditions are given) or when $\sum |a_n| < \infty$; applications to C- and S-functions. Necessary and sufficient conditions for a C-function F and an S-function f to be continuous in $|z| \leq 1$ and satisfy $|f| \leq M < 1$, $\Re(F) \geq m > 0$. Connections with recent work of H. S. Wall on continued fractions and functions $\int_0^1 (1+vu)^{-1} d\beta(u)$.

R. P. Boas, Jr. (Evanston, Ill.).

Geronimus, Ya. L. On asymptotic properties of polynomials which are orthogonal on the unit circle and on some properties of positive harmonic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 123–144 (1950). (Russian)

This paper presents an account of a group of asymptotic properties of polynomials orthogonal on the unit circle; these are mostly due to the author [cf. Mat. Sbornik N.S. 15(57), 99–130 (1944); 23(65), 77–88 (1948); same Izvestiya Ser. Mat. 12, 3–14 (1948); these Rev. 7, 63; 10, 190; 9, 429, and the paper reviewed above]. Several new results are given. Nine types of asymptotic behavior are summarized in a table, giving in most cases two conditions, one in terms of the parameters a_n and one in terms of $\sigma'(\theta)=p(\theta)$ [for notation see the preceding review]. A second table gives seven properties of $\sigma(\theta)$ in terms of asymptotic properties of Δ_n; since the real part of $\frac{1}{2}c_0 + \sum_1^\infty c_k z^k$ is a positive harmonic function, this justifies the second part of the title. The transition from table I to table II is made by means of the formula $|a_n| = (1-\Delta_{n+1}\Delta_{n-1}/\Delta_n^2)^{\frac{1}{2}}$. We quote some of the new results from table I: $\lim P_{n+1}^*(z)/P_n^*(z)=1$ uniformly for $|z| \leq 1$ if and only if $a_n \to 0$;

$$\lim \int_0^{2\pi} |P_n^*(e^{i\theta}) - \pi(e^{i\theta})| = 0,$$

with $\pi(z)$ regular in $|z|<1$ and $1/\pi(z)\epsilon H_2$, if

$$\int_0^{2\pi} p(\theta)^{-1} d\theta < \infty;$$

$\lim P_n^*(e^{i\theta}) = \pi(e^{i\theta})$ almost everywhere if $\sum |a_n|^2 \log^2 n < \infty$; $\lim |P_n^*(z)|^{1/n} = 1$ uniformly in $|z| \leq 1$ if $|a_n| \leq \alpha < 1$ and $\lim n^{-1} \sum_0^{n-1} |a_k| = 0$. R. P. Boas, Jr. (Evanston, Ill.).

✱Geronimus, Ya. L. Teoriya ortogonal'nyh mnogočlenov. [Theory of Orthogonal Polynomials]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 164 pp.

This is a collection of results without proofs but with full references to the literature. It is considerably more detailed than Shohat's book [Théorie générale des polynomes orthogonaux de Tchebichef, Mémor. Sci. Math., no. 66, Gauthier-Villars, Paris, 1934], but much less so than Szegö's [Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. 1, 14]; on the other hand,

it contains many results from the literature since 1939. Among the main topics are formal relations, distribution of zeros, connections with continued fractions and with moment problems, asymptotic formulas, various interpolation and quadrature formulas and their convergence, polynomials orthogonal on a curve or over a region; the convergence of the development of a given function is not discussed.

R. P. Boas, Jr. (Evanston, Ill.).

Marden, Morris. On the polynomial solutions of the generalized Lamé differential equation. Proc. Amer. Math. Soc. 1, 492–497 (1950).

This paper is concerned with the location of the zeros of the polynomials $S(z)$ and $V(z)$ which are defined in the following way. In the generalized Lamé differential equation (∗) $P(z)w'' + Q(z)w' + R(z)w = 0$, where $P(z)$, $Q(z)$, and $R(z)$ are real polynomials of degrees p, at most $p-1$, and at most $p-2$, respectively, it is assumed that $P(z)$ has p distinct zeros $a_j = b_j + ic_j$, of which the first q, $0 < q \leq \frac{1}{2}p$, lie in the upper half-plane, the next q are respectively the conjugate imaginaries of the first q, and the remaining $p-2q$ are real. Furthermore, it is assumed that $Q(z)$ has no factor in common with $P(z)$, and that in the development $Q(z)/P(z) = \sum_{j=1}^p A_j e^{i\alpha_j}/(z-a_j)$, $A_j > 0$, $|\alpha_j| < \frac{1}{2}\pi$, $1 \leq j \leq p$. When a Van Vleck polynomial $V(z)$ replaces $R(z)$ in (∗), the corresponding polynomial solution of (∗) is a Stieltjes polynomial $S(z)$ [cf. Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable, Mathematical Surveys, no. 3, Amer. Math. Soc., New York, 1949; these Rev. 11, 101]. With each pair a_j, $\bar{a}_j$, $j=1, 2, \cdots, q$, there is associated an ellipse $E(j, k)$ with center at γ_j on the real axis such that $\gamma_j = b_j + c_j \tan \alpha_j$, with minor axis parallel to the y-axis and of length $m_j = 2|a_j - \gamma_j|$ and major axis of length $(k^{\frac{1}{2}}m_j)$, where k denotes the number of pairs of conjugate imaginary zeros of $S(z)$. Here the author proves the following theorems. (1) Each nonreal zero of $S(z)$ lies in or on at least one of the ellipses $E(j, k)$. (2) Every nonreal zero of $S^r(z)$ lies in or on at least one of the ellipses $E(j, k+r)$, where $S^r(z)$ is the rth derivative of $S(z)$. (3) Every nonreal zero of $V(z)$ lies in or on at least one ellipse $E(j, k+2)$. (4) Every nonreal zero of $V^r(z)$ lies in or on at least one of the ellipses $E(j, k+2+r)$, where $V^r(z)$ is the rth derivative of $V(z)$. E. Frank (Chicago, Ill.).

Dinghas, Alexander. Zur Darstellung einiger Klassen hypergeometrischer Polynome durch Integrale vom Dirichlet-Mehlerschen Typus. Math. Z. 53, 76–83 (1950).

Let L be a linear polynomial in x which is positive in $a \leq x \leq b$. Then the integral $\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}L^{-\alpha-\beta} dx$ can be evaluated in terms of gamma functions, and this evaluation leads to a functional equation for the polynomials (or functions) generated by L^{-s}. The author applies this process to Gegenbauer polynomials. [The reviewer remarks that the results are actually particular instances of a functional relation for hypergeometric functions which has been discussed by many authors; cf. Koschmieder, Acta Math. 79, 241–254 (1947); these Rev. 9, 351.] The results are extended to double integrals. Let D be a triangular domain in the (x, y)-plane, $L_i(x, y) = 0$, $i=1, 2, 3$, the sides of the triangle, L_i nonnegative in D, and let L be any linear function of x and y which is positive in D. The double integral $\iint_D L_1^{\alpha-1} L_2^{\beta-1} L_3^{\gamma-1} L^{-\alpha-\beta-\gamma} dx dy$ can be evaluated and leads to a functional relation between functions generated by L^{-s}. This process is applied to Gegenbauer polynomials and to

certain polynomials in two variables which were investigated by Hermite.   A. Erdélyi (Pasadena, Calif.).

Grosswald, Emil. On a simple property of the derivatives of Legendre's polynomials. Proc. Amer. Math. Soc. 1, 553–554 (1950).

The derivatives of Legendre's polynomial for $x=1$ are computed with the following result:

$$P_n^{(r)}(1) = (n+r)!/2^r r!(n-r)!.$$

The proof is based on the recurrence formula

$$P_n' = nP_{n-1} + xP'_{n-1}$$

and on the formulas arising from this by repeated differentiation. [The result can be easily obtained from

$$P_n^{(r)}(x) = (2^r r!)^{-1}(d^{n+r}/dx^{n+r})(x-1)^n(x+1)^n$$

by using Leibniz's formula.]   G. Szegö.

Bagchi, Haridas, and Chakravarty, Nalini Kanta. On Tschebyscheff's function $T_n(z)$ and its associated equations. J. Indian Math. Soc. (N.S.) 14, 35–42 (1950).

Bagchi, Haridas, and Chakravarty, Nalini Kanta. Some further relations connected with Tschebyscheff's function $T_n(z)$. J. Indian Math. Soc. (N.S.) 14, 43–46 (1950).

Tchebychef polynomials $T_n(z) = \cos(n \cos^{-1} z)$ satisfy a differential equation with respect to z, and a recurrence relation with respect to n. In the first paper, the authors write down the general solutions of these functional relations, and derive many properties of the functions in question. Most of the results are known and can be found in Szegö's Orthogonal Polynomials [Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. 1, 14]. In the second paper the authors obtain four relations connecting Tchebychef and Legendre polynomials.   A. Erdélyi.

Special Functions

Burchnall, J. L., and Lakin, A. The theorems of Saalschütz and Dougall. Quart. J. Math., Oxford Ser. (2) 1, 161–164 (1950).

Using the differential equations satisfied by the generalized hypergeometric series $F(a_r; c_s; x)$, the authors prove the theorems mentioned in the title and assert that the method may be extended to some nonterminating series, and also to functions of argument -1.   N. J. Fine.

Bailey, W. N. On the basic bilateral hypergeometric series $_2\Psi_2$. Quart. J. Math., Oxford Ser. (2) 1, 194–198 (1950).

The main result of this paper is the transformation (*):

$$_2\Psi_2\left[\begin{matrix} e, f \\ aq/c, aq/d \end{matrix}; \frac{aq}{ef}\right]$$

$$= \prod_{m=1}^{\infty}\left[\frac{(1-q^m/c)(1-q^m/d)(1-aq^m/e)(1-aq^m/f)}{(1-aq^m)(1-q^m/a)(1-aq^m/cd)(1-aq^m/ef)}\right]$$

$$\times \sum_{s=-\infty}^{+\infty} \frac{(qa^{\frac{1}{2}})_s(-qa^{\frac{1}{2}})_s(c)_s(d)_s(e)_s(f)_s}{(a^{\frac{1}{2}})_s(-a^{\frac{1}{2}})_s(aq/c)_s(aq/d)_s(aq/e)_s(aq/f)_s}\left(\frac{a^2q}{cdef}\right)^s q^{s^2}.$$

Letting $d=a$ and $c, e, f \to \infty$, one obtains the formula for $\sum_{n=0}^{\infty} a^n q^{n^2}/(q)_n$ from which the Rogers-Ramanujan identities are usually derived. Another application of (*) arises

from the symmetry of the series in c, d, e, f, which leads immediately to a group of transformations of the general $_2\Psi_2$; these the author has developed earlier in the paper, using a formula of Ramanujan which sums the general $_1\Psi_1$ as an infinite product. [(*) can be derived directly from equation (6.1) of the author's paper in the same Quart., Oxford Ser. (1) 7, 105–115 (1936), by letting $b \to \infty$.]
          N. J. Fine (Philadelphia, Pa.).

Jackson, M. On Lerch's transcendant and the basic bilateral hypergeometric series $_2\Psi_2$. J. London Math. Soc. 25, 189–196 (1950).

Lerch's transcendant is defined by

$$f(x, \xi; q, Q) = \sum_{n=-\infty}^{+\infty} (-\xi^2 Q)_n nq^{n^2}x^{2n}.$$

By use of a formula of Bailey [see the preceding review, (*)] $f(x, xq^n; q, q)$ is expressed in terms of ϑ-functions of modulus q^3, for $n = -1, 0, 1, 2, \cdots$. It is then proved that $\varphi(x, \xi; q, q) \equiv \prod_{m=1}^{\infty}(1+q^{2m-1}\xi^{-2})f(x, \xi; q, q)$ is invariant under $x \to \xi^{-1}, \xi \to x^{-1}$. Finally it is shown that $f(x, \xi; 1, q)$ reduces to an infinite product. [Equations (4.3) and (4.4) are incorrectly printed: The factor $(1+x^2q^{2n})$ should be $(1-x^2q^{2n})$.]   N. J. Fine (Philadelphia, Pa.).

Busbridge, Ida W. On the integro-exponential function and the evaluation of some integrals involving it. Quart. J. Math., Oxford Ser. (2) 1, 176–184 (1950).

The object of this paper is to extend the work of Kourganoff [Ann. Astrophysique 10, 282–299, 329–340 (1947); these Rev. 9, 349, 432; cf. also the further literature quoted in these reviews] on the function $E_n(x) = \int_1^{\infty} t^{-n}e^{-xt}dt$. In particular, integrals of the form

$$\int_0^{\infty} e^{-px}x^{\sigma}E_{r_1}(a_1x) \cdots E_{r_k}(a_kx)dx$$

are considered, where p, σ, ν_m, a_m are complex parameters. For these integrals reduction formulas are developed, and special integrals (for $k=1$; for $k=2$ and $p=0$; for $k=2, \sigma=0$, ν_1 and ν_2 integers) are evaluated.   A. Erdélyi.

Poli, L. Sur les sinus d'ordre supérieur. Cahiers Rhodaniens 1, 15 pp. (1949).

This is a largely expository paper on the so-called trigonometric functions of higher order. The author's main purpose is to coordinate several independent investigations of these functions, and in the process he obtains some results believed to be new. A bibliography stretching over a century is appended.   A. Erdélyi (Pasadena, Calif.).

Bateman, H. Some definite integrals occurring in Havelock's work on the wave resistance of ships. Math. Mag. 23, 1–4 (1949).

In this paper [a posthumous paper edited by A. Erdélyi] some integrals of the form $\int_0^{\pi/2}e^{-\beta \sec^2 \varphi}\cos^{2n+1}\varphi \, d\varphi$, formerly studied by Havelock [Proc. Roy. Soc. London. Ser. A. 100, 582–591 (1925)] are expressed by means of confluent hypergeometric functions by the substitution $\beta \tan^2 \varphi = t$. In this connection two apparently different integral expressions are obtained, which ought to be equivalent because of Kummer's relation [J. Reine Angew. Math. 17, 228–242 (1837)]

$$\frac{Z^b}{\Gamma(b)}\int_0^{\infty} s^{b-1}(1+s)^{-a}e^{-Zs}ds = \frac{Z^a}{\Gamma(a)}\int_0^{\infty} s^{a-1}(1+s)^{-b}e^{-Zs}ds.$$

A simple alternative proof of this result is given by means

of Laplace-transformations, approximately in the following way. The left hand integral

$$= \{Z^c/\Gamma(b)\Gamma(c-b)\} \int_0^\infty e^{-Zt}t^{c-b-1}dt \int_0^\infty e^{-Zs}s^{b-1}(1+s)^{-a}ds$$

$$= \{Z^c/\Gamma(b)\Gamma(c-b)\}\mathfrak{L}(t^{c-b-1})\mathfrak{L}\{(t^{b-1}(1+t)^{-a}\}$$

$$= \{Z^c/\Gamma(b)\Gamma(c-b)\}\mathfrak{L}\{t^{c-b-1}*t^{b-1}(1+t)^{-a}\}$$

$$= Z^c \int_0^\infty e^{-Zt}dt \cdot \{1/\Gamma(b)\Gamma(c-b)\} \int_0^t (t-u)^{c-b-1}u^{b-1}(1+u)du$$

$$= \{Z^c/\Gamma(c)\} \int_0^\infty e^{-Zt}t^{c-1}F(a, b; c; -t)dt.$$

Now the parameters a and b are interchangeable. In the latter part of this paper the author discusses the integrals (Havelock)

$$L_\tau = \int_0^{\pi/2} \cos(2\tau\varphi - k\tan\varphi)d\varphi,$$

$$k > 0, \tau = 0, 1, 2, \cdots,$$

$$M_\tau = \int_0^{\pi/2} \sin(2\tau\varphi - k\tan\varphi)d\varphi;$$

$$L_\tau = (-1)^{\tau-1}\pi e^{-k}{}_1F_1(1-\tau; 2; 2k) \quad (\tau \geq 1),$$

$$L_0 = (2\pi)^{-1}e^{-k}.$$

[for the last two equations see Bateman, Trans. Amer. Math. Soc. **33**, 817–831 (1931)], where M_τ is expressed in different forms by means of L_τ. The simplest expression is $M_\tau = -(\pi)^{-1}L_\tau \operatorname{li}(e^k) + R_{\tau-1}(k)$, where $R_{\tau-1}$ is a polynomial of degree $\tau-1$. *S. C. van Veen* (Delft).

Differential Equations

Calamai, Giulio. Il confronto delle approssimazioni successive di Peano-Picard, coll'integrale di una equazione differenziale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 87–92 (1949).

The author establishes some inequalities between the integral of the differential equation

$$x^{(n)} = f[t, x^{(0)}(t), x^{(1)}(t), \cdots, x^{(n-1)}(t)],$$

$0 \leq t < +\infty$, $-\infty < x^{(h)}(t) = d^h x/dt^h < +\infty$, determined by the initial conditions $x^{(h)}(0) = x_0^{(h)}$, $k = 0, 1, \cdots, n-1$, and the functions which approximate that integral according to the method of Picard-Peano. Put $x_0(t) = \sum_{p=0}^{n-1} x_0^{(p)}t^p/p!$ and denote by $x_i^{(0)}(t)$, $x_i^{(h)}(t)$ the ith order approximations of the integral $x^{(0)}(t)$ and of its derivatives $x^{(h)}(t)$. It is demonstrated that if $x^{(0)}(t)$ and its derivatives satisfy the equation for $0 \leq t < +\infty$, if f possesses 1st order continuous nonpositive partial derivatives with regard to $x^{(h)}$ and the approximations of 0th and 1st orders satisfy the inequalities

$$f[t, x_0^{(0)}(t), x_0^{(1)}(t), \cdots, x_0^{(n-1)}(t)] \leq 0 \quad (\geq 0),$$
$$f[t, x_1^{(0)}(t), x_1^{(1)}(t), \cdots, x_1^{(n-1)}(t)] \leq 0 \quad (\geq 0),$$

then for the integral $x^{(0)}(t)$ and its derivatives $x^{(h)}(t)$, the inequalities

$$x_{2i}^{(k)}(t) > x^{(k)}(t) > x_{2i+1}^{(k)}(t) \quad (x_{2i}^{(k)}(t) < x^{(k)}(t) < x_{2i+1}^{(k)}(t))$$

hold. Furthermore, on the hypothesis that f has 1st order nonnegative derivatives, according as

$$f[t, x_0^{(0)}(t), x_0^{(1)}(t), \cdots, x_0^{(n-1)}(t)] \geq 0$$

or

$$f[t, x_0^{(0)}(t), x_0^{(1)}(t), \cdots, x_0^{(n-1)}(t)] \leq 0,$$

it follows that $x^{(k)}(t) > x_i^{(k)}(t)$ or $x^{(k)}(t) < x_i^{(k)}(t)$.
C. Miranda (Naples).

Teghem, J. Sur une méthode d'obtention d'une intégrale de certaines équations différentielles linéaires, à coefficients constants. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 117–127 (1950).

Given a function F, analytic in a region R, and sequences $\{a_\lambda\}$, $\{b_n\}$, and $\{c_n\}$ with $b_n = \sum a_\lambda c_{n-\lambda}$, and with $a_\lambda = 0$ for all $\lambda > \lambda_0$, the author studies $y = \sum c_n D^n F$ as a formal solution of the differential equation $\sum a_\lambda D^\lambda y = \sum b_n D^n F$. If the series for y is uniformly summable $\{E, b\}$ [same Bull. Cl. Sci. (5) **35**, 1042–1053 (1949); these Rev. **11**, 517], then y is in fact a solution. The author then discusses two conditions which guarantee the summability, and obtains a number of results on their applicability and implications.
R. C. Buck (Madison, Wis.).

Bertolini, Fernando. Sugli integrali di una equazione differenziale ordinaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 285–292 = Consiglio Naz. Ricerche. Pubbl. Inst. Appl. Calcolo no. **275** (1950).

Let the function $z(x)$ be defined by the differential equation $d^n z(x)/dx^n = \sum_{s=0}^{n-1} g_s(x)d^s z(x)/dx^s$ and the initial conditions $[d^s z(x)/dx^s]_{x=x_0} = z_0^{(s)}$, $s = 0, \cdots, n-1$. Here the functions $g_s(x)$ are arbitrary and quasi-continuous in a certain open interval containing the point x_0, and the $z_0^{(s)}$ are arbitrary constants such that $z_0^{(0)} \neq 0$. The author determines an interval, containing x_0, in which $z(x)$ has no zeros. In some cases, at least, the result obtained is a definite improvement on previously known results of the same kind. Thus, in the case of one illustrative example, one previous result leads to a zero-free interval of length 0.707, the present result leads to an interval of length 1.468, and the maximum zero-free interval is known actually to be of length 2.003.
L. A. MacColl (New York, N. Y.).

Wintner, Aurel. On the existence of Laplace solutions for linear differential equations of second order. Amer. J. Math. **72**, 442–450 (1950).

Let $f(s) = \int_1^\infty e^{-st}d\alpha(t)$, where $\alpha(t)$ has finite total variation. Then the differential equation $x'' + f(s)x = 0$ has a solution given by $x = 1 + \int_1^\infty e^{-st}d\beta(t)$, where $\beta(t)$ has finite total variation. Further facts are also given. *N. Levinson.*

Hartman, Philip, and Wintner, Aurel. On the derivatives of the solutions of one-dimensional wave equations. Amer. J. Math. **72**, 148–156 (1950).

The equation (1) $x'' + q(t)x = 0$ ($0 \leq t < +\infty$) is considered, where $q(t)$ is real-valued and continuous, and some theorems on whether the solutions of (1) and their derivatives belong to the class (L^2) are given. It is proved that for suitable $q(t)$ two linearly independent solutions of (1), $x(t)$ and $y(t)$, can both belong to the class (L^2), while for no $q(t)$ can both their derivatives be of class (L^2). If $x'(t)$ is of class (L^2) then $y(t)$ cannot be so, or, more generally, if a solution exists for which (2) $\int_0^t x'^2(s)ds = O(t^2)$ holds, then (1) cannot possess two linearly independent solutions in the class (L^2). Some applications of these results to the equation (3) $x'' + \{q(t) + \lambda\}x = 0$ are then given. It is proved that if a $\lambda = \lambda_0$ exists for which (3) has a solution $x(t) \neq 0$ satisfying (2), then (3) is of Grenzpunkt type and also, either $x(t)$ is of class (L^2) or λ_0 is in the essential spectrum of (3). Furthermore, if $-\infty \leq \limsup q(t) < +\infty$ and for $\lambda = \lambda_0$ an integral $x(t)$ of (3) exists for which (4) $\int_0^t x'^2(s)ds = O(t^N)$ for some N, then either $x(t)$ is of class (L^2) or λ_0 is in the essential spectrum of (3). These possibilities are not mutually exclu-

sive and the second must occur if (4) is satisfied by two linearly independent solutions of (3) corresponding to $\lambda = \lambda_0$.

C. Miranda (Naples).

Cambi, Enzo. **The simplest form of second-order linear differential equation, with periodic coefficient, having finite singularities.** Proc. Roy. Soc. Edinburgh. Sect. A. 63, 27–51 (1950).

The author studies the differential equation

$$(1 + 2\gamma \cos x)y'' + p^2 y = 0,$$

where 2γ and p^2 are real parameters. It is noted that there is no loss in generality in assuming that $\gamma > 0$, and the discussion is then separated into a consideration of the three cases: $2\gamma < 1$, $= 1$, > 1. In the first case the substitution $y = e^{iux} \sum_{-\infty}^{+\infty} B_n e^{inx}$ leads to a system of difference equations satisfied by the numbers B_n. This system is solved by using continued fractions. The method of solution described is not available when $2\gamma \geqq 1$, and direct function-theoretic methods are employed in the latter cases. Instructive graphs and a numerical example are supplied.   *W. Leighton*.

�**Trimmer, John Dezendorf.** **Response of Physical Systems.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. ix+268 pp. $5.00.

The contents of this rather unusual book are indicated by the following list of chapter titles: A pattern for systems; Physical systems; First-order systems; Second-order systems, Sinusoidal forcing of linear systems; Higher-order systems; Measuring instruments; Feedback systems; Parametric forcing; Distributed systems; Nonlinear systems. Thus the book is concerned primarily with the fundamental problems of finding the response of a given system to a given driving force, and of designing a system which shall respond in a preassigned way to a given driving force. Emphasis is placed upon the essential unity of the theory, despite the great variety of physical situations to which it can be applied. The exposition is clear and accurate; it includes the discussion of many interesting and instructive examples. The book should be useful to many engineers and physicists.

Mathematically, the book is quite elementary. Only the simplest methods are used in dealing with differential equations, and the systematic development of the mathematical aspects of the subject practically terminates with the part involving linear second order differential equations with constant coefficients. However, in an appendix there is a brief sketch of Laplace transform methods, and some remarks on the significance and value of these methods.

L. A. MacColl (New York, N. Y.).

Gadsden, C. P. **An electrical network with varying parameters.** Quart. Appl. Math. 8, 199–205 (1950).

This is essentially a study of the solutions of the differential equation $(d/dt)(L dq/dt) + R dq/dt + C^{-1}q = 0$, where L, R, C are positive functions, defined for all real values of t, which are continuous, bounded, and bounded away from zero. The equation admits of various physical applications, including the one mentioned in the title. Using a combination of analytical and geometrical reasoning, the author proves several theorems concerning the solutions. It is shown that either all solutions are oscillatory or all solutions are such that the signs of q and dq/dt ultimately remain fixed. If a solution is not oscillatory, q and dq/dt approach zero as t increases indefinitely. Conditions on

L, R, C are given which are sufficient for the solutions to be oscillatory or nonoscillatory, respectively. In the oscillatory case, an estimate is given of the number of oscillations in a prescribed time interval. The stability of solutions is considered briefly, and two conditions are given which are each sufficient for stability.   *L. A. MacColl.*

Zadeh, Lotfi A. **The determination of the impulsive response of variable networks.** J. Appl. Phys. 21, 642–645 (1950).

This paper, which is intended for engineers, is concerned chiefly with the general formal aspects of the problem of solving a linear ordinary differential equation with variable coefficients. In particular, the author discusses expressions for the solution in terms of various auxiliary functions which have simple interpretations in the physical applications. The case in which the coefficients in the differential equation are slowly varying is considered in some detail, and methods for computing the auxiliary functions in this case are indicated.   *L. A. MacColl* (New York, N. Y.).

Amerio, Luigi. **Studio asintotico del moto di un punto su una linea chiusa, per azione di forze indipendenti dal tempo.** Ann. Scuola Norm. Super. Pisa (3) 3 (1949), 19–57 (1950).

Continuing work begun in an earlier paper [Ann. Mat. Pura Appl. (4) 30, 75–90 (1949); these Rev. 11, 723] devoted to the equation $y'' + ay' + \sin y = b$, which is of importance in connection with the synchronization of motors and electronic circuits, the author investigates the more general equation, $y'' = f(y, y')$, where f is periodic in y. Under certain additional assumptions concerning f, it is shown that there are essentially three types of solutions, stable and unstable solutions corresponding to point solutions in the (y, y')-plane, obtained by setting $f(y, 0) = 0$, and periodic solutions in the (y, y')-plane. The type of solution obtained depends strongly upon the parameters in $f(y, y')$ and upon the initial position. A detailed account is given, and it is not possible to discuss the results briefly. Results pertaining to the special case above have been obtained by Tricomi [same Ann. (2) 2, 1–20 (1933)].   *R. Bellman.*

Bulgakov, A. A. **The dynamics of contact synchronization of electrical drives.** Avtomatika i Telemehanika 8, 108–116 (1947). (Russian)

Investigation of a contact phase synchronization scheme suitable for fractional horsepower motors. The dynamics of the regulation are investigated semi-analytically: condition for stability and transient behaviour.   *H. G. Baerwald.*

Tartakovskiĭ, V. **Explicit formulas for local expansions about a nodal point.** Doklady Akad. Nauk SSSR (N.S.) 72, 853–856 (1950). (Russian)

This paper is a continuation of a recent paper [same vol., 633–636 (1950); these Rev. 12, 27]. In the present note the author continues his treatment by infinite matrices and addresses his attention more particularly to the differential matrices in the neighborhood of a point of equilibrium. He also discusses the transformation of such matrices resulting from analytic transformations of the basic variables.   *S. Lefschetz* (Princeton, N. J.).

Bylov, B. F. **On the characteristic numbers of the solutions of systems of linear differential equations.** Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 341–352 (1950). (Russian)

The author proves the following general result concerning the characteristic numbers, in the sense of Liapounoff, of

linear systems with variable coefficients, $dx/dt = A(t)x$: If $A(t)$ approach $B(t)$ as $t \to \infty$, then the characteristic numbers of the two systems coincide. He also discusses the characteristic numbers of some particular systems.

R. Bellman (Stanford University, Calif.).

Aizerman, M. A. On the determination of the safe and unsafe parts on the boundary of stability. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 444–448 (1950). (Russian)

The author discusses the stability of the system governed by the equations $\dot{x}_1 = \sum_{j=1}^{s} a_{1j}x_j + ax_k$, $\dot{x}_i = \sum_{j=1}^{s} a_{ij}x_j$, $i = 2, 3, \cdots, n$. Two examples are given. *R. Bellman.*

Demidovič, B. P. On a critical case of instability in the sense of Lyapunov. Doklady Akad. Nauk SSSR (N.S.) 72, 1005–1008 (1950). (Russian)

The author considers the nth order linear vector system, $dx/dt = (A + B(t))x$, where A is a constant matrix which has the property that $n - 1$ of its characteristic roots have negative real parts with the remaining characteristic root equal to zero, and $B(t) \to 0$ as $t \to \infty$. Under certain restrictions on the elements of $B(t)$, it is shown that the zero solution is stable, i.e., if $\|x(t_0)\| \le d, d = d(t_0, \epsilon)$, then $\|x(t)\| \le \epsilon$ for $t \ge t_0$, $(\|x\| = \sum_{i=1}^{n}|x_i|)$. *R. Bellman.*

Lur'e, A. I. On the character of the bounds of the region of stability of regulating systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 371–382 (1950). (Russian)

In connection with the theory of regulating systems, one encounters differential equations of the form

$$dx_i/dt = \sum_{j=1}^{n} a_{ij}x_j + b_i f(u),$$

where $u = \sum_{k=1}^{n} c_k x_k$, and $f = \sum_{k=1}^{n} d_k u_k$. The stability of the system depends upon the arithmetic properties of the characteristic roots of the matrix obtained from the linear approximation. The difficult cases where one characteristic root is zero, or two characteristic roots have zero real parts and are conjugate complex, with the other characteristic roots possessing negative real parts, are discussed in this paper using the methods of Liapounoff. *R. Bellman.*

Tihonov, A. N. On systems of differential equations containing parameters. Mat. Sbornik N.S. 27(69), 147–156 (1950). (Russian)

Generalizing his previous work [Mat. Sbornik N.S. 22(64), 193–204 (1948); these Rev. 9, 588] the author considers the system (*) $dy_i/dt = f_i(t, y, z), i = 1, \cdots, n; \mu_j dz_j/dt = F_j(t, y, z)$, $j = 1, \cdots, m$. The μ_j are small positive parameters, depending on a parameter μ in such a way that $\lim_{\mu \to 0} \mu_j(\mu) = 0$, $\lim_{\mu \to 0} \mu_{j+1}/\mu_j = 0$ or 1. The degenerate system obtained by setting all $\mu_j = 0$ is assumed to have continuous solutions; it is further supposed that the roots $z_j = \psi_j(t, y)$ of $F_j = 0$ have continuous first partial derivatives, and are stable. The latter condition means that the expression $\sum_j (z_j - \psi_j(t, y))F_j(t, y, z)$ is negative in a suitable deleted neighborhood N of the roots. The principal theorem states that as $\mu \to 0$ the solutions of (*) tend to the corresponding solutions of the degenerate system, assuming that the initial point (t^0, y_i^0, z_j^0) lies in N. The convergence is uniform for $t \ge t_1 > t^0$. *J. G. Wendel* (New Haven, Conn.).

Malkin, I. G. Certain questions on the theory of the stability of motion in the sense of Liapounoff. Amer. Math. Soc. Translation no. 20, 173 pp. (1950).

[Translated from Sbornik Naučnyh Trudov Kazanskogo Aviacionnogo Instituta im. P. I. Baranova, no. 7, 1937.]

This volume gives a clear and detailed treatment of a number of diverse questions on stability of motion, presented in a unified way. Let (*) $dx/dt = X_i(t; x_1, \cdots, x_n)$, $i = 1, \cdots, n$, be a system of differential equations having the origin $x_1 = \cdots = x_n = 0$ as a solution. The problems studied concern conditions under which the zero-solution possesses Liapounoff or asymptotic stability, and the results are obtained by means of "Liapounoff functions," which are functions $V = V(t; x_1, \cdots, x_n)$ vanishing at the origin, definite or semidefinite elsewhere, and having semidefinite total time derivatives of opposite sign, where, of course, dV/dt is calculated from (*). Liapounoff showed that under very general conditions the existence of such a V guarantees the stability of $(0, \cdots, 0)$; mild additional restrictions yield asymptotic stability. In chapters I and II the author investigates the converse proposition, and obtains general conditions under which stability implies the existence of a Liapounoff function; in I, $n = 2$, and t is absent from the right members; in II the equations are linear with uniformly bounded coefficients. In chapter III the method of Liapounoff functions is applied to cases in which the right members of (*) can be replaced by their leading terms; Liapounoff studied this case, but by a different method. Chapter IV deals with more complicated problems, where the equations of first approximation do not yield sufficient information. The two concluding chapters are concerned with stability in certain special cases. Chapter V treats the case where the X_i do not contain t explicitly, the leading terms are linear, and the characteristic equation of the system of first approximation has a double zero. Chapter VI discusses periodic solutions of (*) when the X_i are periodic in t.

J. G. Wendel (New Haven, Conn.).

Malkin, I. G. Oscillations of systems with one degree of freedom close to systems of Lyapunov. Amer. Math. Soc. Translation no. 22, 63 pp. (1950).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 561–596 (1948); these Rev. 10, 457.

Mitropol'skiĭ, Yu. A. Slow processes in nonlinear oscillating systems with many degrees of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 139–170 (1950). (Russian)

Bogoliuboff [in an unavailable paper] developed an approximate method to investigate single frequency oscillations of a nonlinear system with many degrees of freedom. The author extends this method to the case of a nonlinear system with the independent variable t appearing in the form ϵt, where ϵ is small, and also in the form of $\theta(t)$, where $d\theta/dt = \nu(\epsilon t) \ge 0$ and $\nu(\tau)$ is a continuous function of τ. Thus the terms involving t change slowly with t when ϵ is small except for the linear part of $\theta(t)$. Besides the formal and mathematical development of the method, examples are given. *N. Levinson* (Cambridge, Mass.).

Serbin, H. Periodic motions of a nonlinear dynamic system. Quart. Appl. Math. 8, 296–303 (1950).

It is shown that the equation $x'' + f(x)x' + g(x) = 0$ has a unique periodic solution under quite general conditions on the continuous functions f, g. For positive x these conditions are: $f(x) < 0$ for $0 < x < x_1$, $f(x) > 0$ for $x > x_1$, $g(x) > 0$ for $x > 0$, $F(\infty) > 0$, $F(\infty)G(\infty) = \infty$, where $F(x) = \int_0^x f(u)du$, $G(x) = \int_0^x g(u)du$. For negative x the conditions are: $f(x) < 0$ for $x_1' < x < 0$, $f(x) > 0$ for $x < x_1'$, $-g(x) > 0$ for $x < 0$, $-F(-\infty) > 0$, $-F(-\infty)G(-\infty) = \infty$. Bounds for the amplitude of the periodic solution are obtained. The theorem

includes results of Levinson and Smith [Duke Math. J. 9, 382–403 (1942); these Rev. 4, 42]. *J. G. Wendel.*

Urabe, Kojûrô. On the existence of periodic solutions for certain non-linear differential equations. Math. Japonicae 2, 23–26 (1950).

The author shows that the differential equation

$$x'' + g'(x)x' + f(x) = e(t)$$

has at least one periodic solution if (I) f and g are differentiable, (II) e is periodic, and (III) there exist positive constants p and q such that $p > q^2$, $p + 4q^2 \geqq qG(x) \geqq F(x) \geqq p$, where $F(x) = f(x)/x$, $G(x) = g(x)/x$. *J. G. Wendel.*

Reeb, Georges. Sur l'existence de solutions périodiques de certains systèmes différentiels perturbés. Arch. Math. 2, 205–206 (1950).

Using classical theorems of topology the author shows that certain fields of vectors contain closed trajectories and applies his result to a slightly perturbed system of an odd number of simple harmonic oscillators. [Actually from the point of view of differential equations the slightly perturbed system can be handled effectively by small parameter methods. In the relaxation case of large departure from the linear system, the present method does not seem to be applicable.] *N. Levinson* (Cambridge, Mass.).

Wazewski, T. Sur la coïncidence asymptotique des intégrales de deux systèmes d'équations différentielles. Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1949, 147–150 (1950).

If $Y(x)$ and $Z(x)$ in the differential equations

$$dY/dx = F(x, Y), \quad dZ/dx = G(x, Z)$$

are vectors of n components, and the equations are such as have solutions over the interval $-\infty < x < \infty$, two such solutions $\bar{Y}(x)$ and $\bar{Z}(x)$ are commonly said to be asymptotically equivalent for $x = +\infty$ if $\lim_{x \to +\infty} |\bar{Y}(x) - \bar{Z}(x)| = 0$. This notion is not invariant under regular transformations. The paper gives two definitions of asymptotic equivalence which are invariant. Of these, one is roughly the following. Let $E_0(Y, r)$ denote the zone that is filled by the integrals $Y(x)$ which pass through the neighborhood of radius r of a point (x_0, Y_0). Two integrals $\bar{Y}(x)$, $\bar{Z}(x)$ are designated to be asymptotically equivalent if for every $r > 0$, $\bar{Y}(x)$ is in $E_0(Z, r)$ and $\bar{Z}(x)$ is in $E_0(Y, r)$, for all sufficiently large x. The second definition is of a similar character but is not equivalent to this. *R. E. Langer* (Madison, Wis.).

Obmoršev, A. N. Investigation of phase trajectories at infinity. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 383–390 (1950). (Russian)

The Poincaré system (1) $dx/dt = P(x, y)$, $dy/dt = Q(x, y)$ is investigated for a limit-cycle at infinity in the following manner. The phase plane is projected onto a hemisphere H of radius one tangent to the plane at the origin; then H is projected orthogonally on the plane. If ρ, θ are the polar coordinates, this yields the transformation $x = \rho(1-\rho^2)^{-\frac{1}{2}}\cos\theta$, $y = \rho(1-\rho^2)^{-\frac{1}{2}}\sin\theta$, and it leads from (1) to (2) $d\rho/d\theta = \rho(1+\rho)(1-\rho)^{-m}\Phi(\rho, \theta)/\Psi(\rho, \theta)$ with Φ, Ψ analytic and not divisible by $1-\rho$. Infinity is now imaged into C, the circle of radius one. If $\Psi \neq 0$ on C and $m > 0$, C is a limit-cycle. If $m = 0$, C is not a trajectory. Finally if $m < 0$, C is a closed curve without contact. Stability and the existence of limit cycles approaching C are discussed.

Application is made to the study of the trajectories of the system (1) corresponding to the following equation derived from the coupling of a series generator with an independently excited motor: $\ddot{x} - \mu((1+x^2)^{-1} - \nu)\dot{x} + x = 0$. [References, besides the classical writings of Poincaré, Bendixson, and Liapounov: Petrovski, Rec. Math. [Mat. Sbornik] (1) 41, 107–155 (1934); von Mises, Compositio Math. 6, 203–220 (1938); Lefschetz, Lectures on Differential Equations, Annals of Mathematics Studies, no. 14, Princeton University Press, 1946, p. 142; these Rev. 8, 68.] *S. Lefschetz.*

Barbašin, E. A. On the existence of smooth solutions of some linear partial differential equations. Doklady Akad. Nauk SSSR. (N.S.) 72, 445–447 (1950). (Russian)

Consider the system (1): $\dot{x} = X(x)$, where $x = (x_1, x_2, \cdots)$ is a point in Euclidean n-space E_n and $X = (X_1, X_2, \cdots)$ are functions of class C_r. A closed set $F \subset G \subset E_n$ is called a "section" of the domain G if any trajectory Γ of (1) meets F in just one point. The system of trajectories is called "unstable" if for any trajectory Γ and any open set H such that $\bar{H} \subset G$ the moving point leaves H both for $t \to +\infty$ and for $t \to -\infty$. The system of trajectories has an "improper saddle" if a sequence of point triads (p_n, q_n, r_n) exists such that p_n, q_n, $r_n \varepsilon \Gamma_n$, q_n lies between p_n and r_n, $p_n \to p$, $r_n \to r$, p, $r \varepsilon G$, but q_n does not converge in G. Theorem 1. If the system of trajectories is unstable and has no improper saddle, "if the time length" of all the trajectories in G has a positive lower bound, then: (1) A section S of class C_r exists in G; (2) a first integral of (1) of class C_r exists which is not constant in any open subset; (3) for each $\psi(x) \varepsilon C_r$ defined on S and each $\varphi(x) \varepsilon C_r$ defined in G, a $v(x) \varepsilon C_r$ exists such that $\sum X_i \cdot (\partial v/\partial x_i) = \varphi$ in G, $v = \psi$ on S. A theorem of Kamke asserts the existence of a regular first integral for $n = 2$; for $n > 2$ this is not true in general but the following result holds. Theorem 2. If G is homeomorphic to E_3 and if the trajectories of (1) satisfy the assumptions of theorem 1, there exist two first integrals u_1, u_2 such that the rank of the Jacobian matrix is 2 in G. Suppose now that O is an asymptotically stable singular point. Theorem 3. Let $R^2 = \sum x_i^2$ and suppose that in the neighborhood of O, $\sum X_i^2 \leqq k^2 R^2$ and $\varphi(x) \geqq m > 0$. If $q = \min(r, [m/k] - 2)$ ([ ] = integral part), in the domain of attraction of O there is a $v \varepsilon C_q$ such that $\sum X_i \cdot (\partial v/\partial x_i) + \varphi \cdot v = 0$ and which coincides with a given function ψ on a given section S of the domain. As a corollary follows the existence of a function of Liapounoff of class C_r, independent of t, which is a stronger result than one previously proved by the reviewer [Ann. of Math. (2) 50, 705–721 (1949); these Rev. 11, 721].

J. L. Massera (Montevideo).

Schwesinger, G. On one-term approximations of forced nonharmonic vibrations. J. Appl. Mech. 17, 202–208 (1950).

Using a least squares procedure the author obtains a one-term simple-harmonic approximation to a harmonically excited nonlinear differential equation of the second order. Examples are given and the method compared with other approximate procedures. *N. Levinson.*

Tschakaloff, Ljubomir [Čakalov, L.]. Über die Riccatischen Differentialgleichungen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 37, 149–195 (1941). (Bulgarian. German summary)

Spenke, Eberhard. Die Diffusionstheorie der positiven Säule mit Berücksichtigung der stufenweisen Ionisierung. Z. Physik **127**, 221–242 (1950).

The Schottky [Phys. Z. **25**, 342–348, 635–640 (1924)] diffusion theory of the positive column considers electron concentration built up by a one-step ionization process, and the present paper is concerned with electron concentration induced by a two-step ionization process. There is shown to be a quantitative but no qualitative difference in the results of the two processes. Mathematically the problem is to find a solution of the system

$$(*) \qquad \frac{d^2N}{dr^2} + \frac{\delta}{r}\frac{dN}{dr} + \tau N + \alpha N^2 = 0,$$

$$N(0) = N_0, \quad N'(0) = 0, \quad N(R) = 0,$$

where τ, α are parameters and N_0, R are fixed numbers. Two cases, $\delta = 0$ and $\delta = 1$, are treated. A solution of $(*)$ omitting condition $N(R) = 0$ is first obtained. The author then considers α near zero or very large and discusses the problem of adjusting τ so $N(R) = 0$ is satisfied. Here N represents the electron concentration and the Schottky case corresponds to $\alpha = 0$. *F. G. Dressel* (Durham, N. C.).

Mitrinovitch, D. S. Sur une équation différentielle linéaire du second ordre intervenant dans un problème de physique mathématique. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 189–193 (1949). (Serbian. Russian and French summaries)

The differential equation given by Milankovitch [Ann. Physik (4) **43**(348), 623–638 (1914)] may be written in the form

$$(1) \qquad \frac{d^2\epsilon}{dx^2} - 3\frac{a'}{a}\frac{d\epsilon}{dx} + \left(3\frac{a'^2}{a^2} - \frac{a''}{a}\right)\epsilon = 0,$$

where $a(x)$ is an arbitrary function. Then, this function is a particular solution of (1), which is a special case of an equation given by Görtler [Z. Angew. Math. Mech. **22**, 233–234 (1942)]. The more general result is: the differential equation (2) $FHy'' + (F'H + FI + GH)y' + (G'H + GI)y = 0$ may be reduced to a system $Fy' + Gy = z$, $Hz' + Iz = 0$, which may be integrated. *W. Jardetzky* (New York, N. Y.).

Marčenko, V. A. Concerning the theory of a differential operator of the second order. Doklady Akad. Nauk SSSR. (N.S.) **72**, 457–460 (1950). (Russian)

Given a differential operator L where $L(u) = u'' - q(x)u$ on a finite or semi-infinite interval with suitable boundary conditions, then it is well known that a spectral function $\rho(\lambda)$ is determined. The author shows that $\rho(\lambda)$ determines $q(x)$ uniquely. He then considers the inverse spectral problem and generalizes results of Borg [Acta Math. **78**, 1–96 (1946); these Rev. **7**, 382] and Levinson [Mat. Tidsskr. B. 1949, 25–30 (1949); these Rev. **11**, 248] to the case of a discrete spectrum in the singular case of L. He next considers the determination of the potential by the asymptotic phase obtaining results of Levinson [Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 9 (1949); these Rev. **10**, 710]. [The author does not use the hypothesis $q(x) \geq 0$ or the alternative hypothesis $\int_1^{\infty} x^2|q(x)|dx > 0$ used by Levinson. It appears, however, that he needs some such hypothesis to validate his argument about the behavior of $g(z) = M_1(z)/M_2(z)$ near $z = 0$ by handling the possibility that $M_2(0)$ vanishes.] *N. Levinson* (Cambridge, Mass.).

Titchmarsh, E. C. On the discreteness of the spectrum associated with certain differential equations. Ann. Mat. Pura Appl. (4) **28**, 141–147 (1949).

The author gives a new proof for the result of Weyl that the spectrum of $\Phi'' + (\lambda - q(x))\Phi = 0$, $-\infty < x < \infty$, is discrete if $q(x) \to \infty$ as $|x| \to \infty$. He then uses his method to prove the corresponding result, due to Friedrichs, for $\Delta\Phi + (\lambda - q(x, y))\Phi = 0$ in the whole (x, y)-plane if $q \to \infty$ as $x^2 + y^2 \to \infty$. *N. Levinson* (Cambridge, Mass.).

✱Levitan, B. M. Razloženie po sobstvennym funkciyam differencial'nyh uravnenii vtorogo poryadka. [Expansion in Characteristic Functions of Differential Equations of the Second Order]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 159 pp.

Expansion in a finite interval; Parseval's equation; The spectrum of a differential operator of the second order; Examples; Refinement of expansion theorems in the case $q(x) \subset L_{12}(0, \infty)$; The resolvent; The interval $(-\infty, \infty)$; Appendix I, Helly's theorem; Appendix II, The Stieltjes transform formula. *Table of contents.*

Barenblatt, G. I. On a method of solution of the equation of heat conduction. Doklady Akad. Nauk SSSR (N.S.) **72**, 667–670 (1950). (Russian)

The author considers the equation $y'' + \lambda q(x) = 0$, where $q(x)$ satisfies the conditions (1) $q(x) \geq 0$ and can be zero only for $x = 0$, (2) $q(x)$ has continuous second derivatives everywhere, and (3) near the origin $q(x) = a_0 x^m[1 + R(x)]$, $R(x) \to 0$ for $x \to 0$, $m \geq 0$. Using a method introduced by Levitan [see the preceding review] and a result due to Titchmarsh [Eigenfunction Expansions . . . , Oxford, 1946; these Rev. **8**, 458] the following theorem is proved: If $q(x)$ satisfies the conditions (1), (2), (3), $q' = O(q^c)$, $0 < c < \frac{4}{3}$, q'' does not change sign and $\int_0^{\infty} q^{\frac{1}{2}}dx$ diverges, then the spectral operator $q^{-1}d^2/dx^2$ is continuous for $\lambda \geq 0$. If $q(x)$ satisfies the conditions of the above theorem, then it is shown that the equation $\partial^2T/\partial x^2 = q(x)\partial T/\partial t$, subject to the conditions $T(x, 0) = 0$, $T(0, t) = \phi(t)$, has a solution expressed as an integral depending on a solution of the ordinary differential equation. A similar representation is given for the heat problem in which the boundary condition is replaced by $\partial T(0, t)/\partial x = -f(t)$. Solutions are explicitly exhibited for an example in which $q(x) = x^m$. For $m = 0$, the well-known solutions are obtained. *C. G. Maple* (Washington, D. C.).

Rapoport, I. M. On a variational problem in the theory of ordinary differential equations with boundary conditions. Doklady Akad. Nauk SSSR (N.S.) **73**, 889–890 (1950). (Russian)

The author considers the equation $x''(t) + \lambda p(t)x(t) = 0$, $0 \leq t \leq T$, with boundary conditions $x(0) = x(T) = 0$, where $p(t)$ is nonnegative in the interval $(0, T)$ and satisfies the normality condition $T^{-1}\int_0^T p(t)dt = 1$. It is shown that for each n, $\lambda_n > 4n^2/T^2$ and that this inequality cannot be improved. Further, for each n, the eigenvalues λ_n can take any value greater than $4n^2/T^2$. *C. G. Maple.*

Zwirner, Giuseppe. Alcuni teoremi sulle equazioni differenziali dipendenti da un parametro. Ann. Triestini. Sez. 2. (4) **2**(18), 5–34 (1949).

The author states and proves approximately a dozen theorems concerning the existence and uniqueness of solutions of eigenvalue problems of the forms: (I) $y'(x) = \lambda f(x, y(x))$, $y(x_0) = y_0$, $y(x_1) = y_1$; (II) $y'(x) = f(x, y(x), \lambda)$, $y(x_0) = y_0$,

$y(x_1) = y_1$; (III) $y'(x) = \lambda f(x, y(x))$, $y(x_0) = y_0$, $y'(x_1) = \alpha$; (IV) $y''(x) = \lambda f(x, y(x), y'(x))$, $y(a) = \alpha$, $y(x_0) = y_0$, $y(b) = \beta$ $(a < x_0 < b)$. The character of the results obtained is indicated by the following typical theorem. Let the function $f(x, y)$ be everywhere finite in the rectangle $R: x_0 \leq x \leq x_1$, $c \leq y \leq d$, and be measurable with respect to x and continuous with respect to y. Let there exist functions $\chi(x)$ and $\varphi(x)$ which are summable in $x_0 \leq x \leq x_1$, and are such that $\chi(x) \geq 0$, $\int_0^{x_1} \varphi(x) dx > 0$, $|f(x, y)| \leq \chi(x)$, $f(x, y) \geq \varphi(x)$. Then, if y_0 and y_1 are numbers between c and d such that

$$|y_1 - y_0| \int_{x_0}^{x_1} \chi(x) dx \leq (d - y_0) \int_{x_0}^{x_1} \varphi(x) dx,$$

$$|y_1 - y_0| \int_{x_0}^{x_1} \chi(x) dx \leq (y_0 - c) \int_{x_0}^{x_1} \varphi(x) dx,$$

the problem (I) possesses at least one solution $(\lambda_0, y_0(x))$ with $y_0(x)$ absolutely continuous in the interval (x_0, x_1).

L. A. MacColl (New York, N. Y.).

Moisil, Gr. C. Les systèmes d'équations aux dérivées partielles et les nombres hypercomplexes. An. Acad. Repub. Pop. Romàne. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 2, no. 14, 409–424 (1949). (Romanian. Russian and French summaries)

The system of partial differential equations can be written in matrix notation as $D\phi = 0$. When D is written in the form $\sum_k \gamma_k \partial/\partial x_k$, the coefficients γ_k are certain constant matrices which belong to a system of hypercomplex numbers. The first example is the equations of motion of an incompressible fluid with a conservative field of vorticity: the hypercomplex numbers are quaternions. In the case of the Maxwell equations, one has biquaternions (that is, quaternions with complex elements). The equations of the two-dimensional theory of elasticity lead to the algebra generated by a primitive root of the equation $1 + 2j^2 + j^4 = 0$. The same system of hypercomplex numbers is obtained in the discussion of the slow, two-dimensional, and steady motion of an incompressible viscous fluid.   A. Erdélyi.

Ciorănescu, Nicolae. On the symbolic or analytic structure of the general integral of some partial differential equations of the second order with constant coefficients. Acad. Repub. Pop. Romàne. Bul. Şti. A. 1, 439–448 (1949). (Romanian. Russian and French summaries)

The partial differential equation $D^2 u + u_{tt} = 0$, in which D is a linear differential operator in $x_1, \cdots, x_n$ with constant coefficients, has the general solution

$$u(x_1, \cdots, x_n, t) = (\cos tD) f(x_1, \cdots, x_n) + (\sin tD) g(x_1, \cdots, x_n).$$

This symbolic solution may be interpreted analytically by means of a Taylor expansion. There is a corresponding statement for the hyperbolic equation, and a few examples.   A. Erdélyi (Pasadena, Calif.).

Cooper, J. L. B. The application of multiple Fourier transforms to the solution of partial differential equations. Quart. J. Math., Oxford Ser. (2) 1, 122–135 (1950).

The author's aim is to circumvent some of the difficulties encountered in solving the initial value problem for a partial differential equation with constant coefficients by means of the ordinary Fourier transformation. He considers the wave equation $\partial^2 f/\partial x_1^2 + \cdots + \partial^2 f/\partial x_k^2 = \partial^2 f/\partial t^2$ with initial conditions $f(x, 0) = \psi(x)$, $f_t(x, 0) = \phi(x)$. The "truncated" Fourier transform $F_D(u, t) = \int_D f(x, t) e^{iu \cdot x} dx$ of F over a bounded

domain of x-space is introduced. To extend the validity of the inversion formula use is made of spherical Abelian means. One obtains

$$(1) \qquad f(x, t) = \lim_{b \to \infty} (2\pi)^{-k} \int e^{-ip - iu \cdot x} F_D(u, t) du$$

for any continuous f, where $\rho^2 = u_1^2 + \cdots + u_k^2$. Also, F_D satisfies an ordinary differential equation in t, the solution of which leads by means of (1) to an expression for f in terms of ψ, ϕ, and of values of f and its normal derivative on the boundary of D. For odd $k = 2m + 1$ it is shown that the resulting expression for f is independent of the boundary values and that it can be expressed in terms of derivatives of certain means of ϕ and ψ, provided ϕ has derivatives of order m and ψ of order $m + 1$. The author's method enables him to give a set of conditions on the initial data, which are "transmitted" to the corresponding solution: If $f_t(x, t)$ has x-derivatives of order s and $f(x, t)$ has x-derivatives of order $s + 1$ for $t = 0$, which are of summable square over any bounded set, then the same is true for any t.   F. John.

Frank, F. C. Radially symmetric phase growth controlled by diffusion. Proc. Roy. Soc. London. Ser. A. 201, 586–599 (1950).

The diffusion equation with radial symmetry is considered in two and in three dimensions. In both cases, particular solutions of the equation are found which represent the growth of a new phase starting from zero radius in a uniform medium and maintaining constant "equilibrium" conditions at the surface of the growing new phase. The diffusion equation is reduced to the form $\partial^2 \phi/\partial s^2 = -(s/2 + 2/s) \partial\phi/\partial s$ by the introduction of a dimensionless "reduced radius" $s = r D^{-1/2} t^{-1/2}$, where r is the radius, D the diffusivity, and t the time, and $\phi(r, t) = \phi(s)$. This leads to a solution of the form $\phi - \phi_\infty = A F(s)$, where ϕ_∞ is the value of ϕ at infinity, A is a constant to be determined, and

$$F(s) = s^{-1} e^{-s^2/4} - \tfrac{1}{2} \pi^{1} [1 - \text{erf} (s/2)].$$

The function $F(s)$ is given both graphically and numerically. The constant A is determined by equating the diffusion flux to the rate of expulsion of the diffusing entity at the surface of the growing new phase. When this value of A is substituted into the above solution, the reduced radius s is shown as a function of $(\phi_s - \phi_\infty)/q$, where q is the amount of the diffusing entity expelled per unit volume of the new phase. This relationship is also shown graphically and numerically. The function $F(s)$ is given as a series convenient for use when s is small and also in an asymptotic expansion which is preferable for large values of s. The above results are applied to growth controlled by the diffusion of heat, or material, or both together, and the effect of the incorporation of impurity into the new phase is discussed.   C. G. Maple (Washington, D. C.).

Jost, W. Bemerkung zur mathematischen Behandlung komplizierter Diffusionsprobleme. Z. Physik 127, 163–167 (1950).

The author considers the differential equation for one-dimensional diffusion in which the diffusion coefficient is a function of the concentration. Although this equation is not integrable in general, the diffusion coefficient may be determined as a function of the concentration by use of the Boltzmann method [Ann. Physik Chemie N.S. 53(289), 959–964 = S.-B. Math. Phys. Cl. Bayer. Akad. Wiss. 24, 211–217 (1894)], subject to certain stated conditions. This

method may be applied to more general problems provided that the boundary conditions (not necessarily linear) do not explicitly contain time. It is further pointed out that in certain cases in which the boundary conditions do explicitly contain time, the general approach of the Boltzmann method enables one to make certain qualitative deductions concerning the relation between the diffusion coefficient and the concentration. *C. G. Maple* (Washington, D. C.).

Fourès-Bruhat, Yvonne. Un théorème d'existence sur les systèmes d'équations aux dérivées partielles quasi linéaires. C. R. Acad. Sci. Paris **231**, 318–320 (1950).

The system is hyperbolic of the second order in four independent variables. It has degree one in the second derivatives and is solvable for the derivatives of the type $\partial^2 w/\partial x^2$, where w is an unknown and x is the same for all unknowns. Cauchy's existence theorem is extended to the case in which the coefficients and initial determinations have bounded, continuous derivatives of order four or five satisfying Lipschitz conditions. *J. M. Thomas*.

Schröder, Kurt. Das Problem der eingespannten rechteckigen elastischen Platte. I. Die biharmonische Randwertaufgabe für das Rechteck. Math. Ann. **121**, 247–326 (1949).

Essentially the problem is to determine the solution of (1) $\Delta^2 \psi = 0$ with assigned values (2) ψ and $\partial \psi/\partial n$ on the boundary of a rectangle. With $u = \partial \psi/\partial x$ and $v = \partial \psi/\partial y$ and (3) $\gamma = \partial u/\partial x + \partial v/\partial y$, the original problem is included in the determination of u and v from the fact that γ is harmonic and that $\partial u/\partial y = \partial v/\partial x$ subject to (4): prescribed boundary values for u and v on S. To achieve this reformulation the writer assumes that except for jump discontinuities at the vertices of S, ψ and $\partial \psi/\partial n$ are of class C^3 and C^2, respectively. Using the superposition principle the solution is reduced to the case of boundary functions in (4) which either vanish or are symmetric or antisymmetric about the midpoints of the sides. The method is to write down formal series for u and v in products of trigonometric and hyperbolic functions with 1, or x, or y and use the conditions (4) to determine the coefficients in terms of the coefficients in the Fourier expansion of the known boundary functions. This procedure goes back to Mathieu and involves the solution of an infinite system of linear equations. The Neumann series is used for solving these equations, though for practical approximation other methods are desirable [cf. Bourgin, Amer. J. Math. **61**, 417–439 (1939)]. Let $S^{(n)}$ be a sequence of rectangles similar to and approaching S. The author shows that if the data are continuous with piecewise continuous derivatives of u and v (the jump discontinuities are restricted to the vertices of the rectangle S), then the solution is unique in a certain aggregate of competing functions, provided that the order of growth of (a) certain first derivatives of u and v, and (b) of the mean values of u and v on S^n be not too great as n tends to ∞. *D. G. Bourgin*.

Halilov, Z. I. Solution of the general problem of the deflection of a simply supported elastic plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 405–414 (1950). (Russian)

The deflection $w(x, y)$ of a thin, simply supported, elastic plate which occupies an open set S with boundary L satisfies on S the partial differential equation $D\Delta\Delta w = p$, and on L the boundary conditions

$$\mu\Delta w + (1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 w}{\partial x \partial y}\cos\theta\sin\theta\right] = 0,$$

where $D > 0$ and $0 \leq \mu < 1$ are physical constants,

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

$p(x, y)$ is the intensity of the normal load on the plate, and θ is the angle between the outer normal to the curve L and the x-axis. The present paper deals with the above boundary value problem for a bounded simply connected S, assuming that the simple closed curve L has a differentiable curvature, and that the derivative of the curvature with respect to arc length satisfies a Hölder condition. Upon assuming that p has continuous first partial derivatives on S, it is shown that the boundary value problem has one and only one solution among the class of real-valued functions $u(x, y)$ defined on $S+L$, with continuous second derivatives on $S+L$, and such that $\partial\Delta u/\partial x$ and $\partial\Delta u/\partial y$ are continuous on $S+L$. The procedure employed is to replace the boundary value problem by an equivalent singular integro-differential equation. The exterior boundary-value problem (where S is replaced by the exterior S^* of the simple closed curve L, and an additional condition is imposed on $w(x, y)$ at infinity) is also considered.

J. B. Dias (College Park, Md.).

John, Fritz. On linear partial differential equations with analytic coefficients. Unique continuation of data. Comm. Pure Appl. Math. **2**, 209–253 (1949).

Consider the linear mth order partial differential equation (1) $L(y) = f$, where f and the coefficients of L are analytic in a suitable region. Let M be an analytic manifold. The author is concerned with the questions: (a) the region in which Cauchy data on M determine the solution of (1) uniquely; and (b) whether the Cauchy data can be assigned arbitrarily. Let $A^{i_1 \cdots i_m}(P)p_{i_1} \cdots p_{i_m} = Q(P)$ be the characteristic form at the point P. Then (3) $Q(P) = 0$ is the characteristic equation. The author interprets $\{p_j\}$ as homogeneous coordinates in a projective $(n-1)$-space, so that the "characteristic directions" are "points" on the characteristic surface. The crucial factor for (a) and (b) is whether the datum surface is characteristic at some point P. If not, then in the author's terminology it is "free" at P. A surface is space-like at P if it is free and contains no lower-dimensional free surface; otherwise, it is time-like. For (a) the conclusion is essentially: If one can homotopically deform M keeping the boundary fixed so that all of the intermediate surfaces have free points only, then a solution of (1), sufficiently continuous, in the region spanned by the initial and final manifolds is uniquely determined by the Cauchy data on M. For (b) the data on time-like surfaces cannot be prescribed arbitrarily. However, in a special case with a strengthening of the requirements for a space-like manifold it is shown that the data can be assigned at will. The method of demonstration consists in showing that, for time-like manifolds, $K(t_1, \cdots, t_r) = \int_{R(t_1, \cdots, t_r)} uw$ is analytic, where u satisfies (1) with $f = 0$, w is an arbitrary analytic function, and $R(t_1 \cdots t_r)$ refers to a family of free analytic manifolds. Thus the values of K on a containing $(r+1)$-dimensional manifold can be obtained by continuation, whence, in view of the arbitrariness of w, u must admit the same sort of continuation even though of course it need not be analytic. In short, then, u cannot be arbitrarily prescribed on the containing manifold.

D. G. Bourgin (Urbana, Ill.).

Functional Analysis, Ergodic Theory

Yvon, Jacques. Forme générale des opérateurs d'Hermite différentiels. Cahiers de Physique no. **33**, 25–30 (1948).

Let P be the operator $-ihd/dx$, Q the operation of multiplication by x, and T a linear differential operator with polynomial coefficients. Then by virtue of the rule $[P, Q] = -ihI$, T can be expressed as $\sum \alpha_{ij} Q^i P^j$ for suitable complex numbers α_{ij}. The polynomial $\alpha_{ij} x^i y^j$ is called the generating function of T, and equals $\exp(-ih^{-1}xy)T(\exp(ih^{-1}xy))$. It is shown that if T is Hermitian (formally), then its generating function f is determined by the real part g of f as follows: $f = g + \tanh(-\frac{1}{2}ihV)g$, where V denotes the differential operator $\partial^2/\partial x \partial y$. The situation is analogous when more variables are involved.

I. E. Segal (Chicago, Ill.).

Zaanen, A. C. Transformations in Hilbert space which depend upon one parameter. Mathematica, Zutphen B. **13**, 13–22 (1944). (Dutch)

Expository paper.

Fage, M. K. Idempotent operators and their rectification. Doklady Akad. Nauk SSSR (N.S.) **73**, 895–897 (1950). (Russian)

Dans un espace de Hilbert $\mathfrak{E}$, soit J un opérateur borné vérifiant $J^2 = J$; l'auteur démontre tout d'abord quelques propriétés élémentaires des opérateurs de ce genre. Soit ensuite J_n une suite de tels opérateurs; on dit que c'est une "décomposition de l'unité" si: (a) $J_m J_n = 0$ pour $m \neq n$; (b) $x = \sum J_n x$ pour tout $x \epsilon E$ (la série étant fortement convergente). On dit qu'une telle décomposition est "rectifiable" s'il existe une décomposition de l'unité en projecteurs (au sens usuel dans la théorie des espaces de Hilbert) P_n, et un opérateur S borné et inversible, tels que $J_n = S P_n S^{-1}$; pour qu'il en soit ainsi, il est nécessaire et suffisant qu'il existe une constante C $(0 < C < +\infty)$ avec $C^{-1}\|x\|^2 \leq \sum \|J_n x\|^2 \leq C\|x\|^2$ pour tout $x \epsilon E$. Enfin, l'auteur indique quelques relations de ses résultats avec la théorie des systèmes biorthogonaux.

R. Godement (Nancy).

Kaplansky, I. Quelques résultats sur les anneaux d'opérateurs. C. R. Acad. Sci. Paris **231**, 485–486 (1950).

Let H be a Hilbert space and let $\mathfrak{A}$ be a self-adjoint weakly closed ring of operators in H. It has been shown by Dixmier [Ann. Sci. École Norm. Sup. (3) **66**, 209–261 (1949); these Rev. **11**, 370] that if $\mathfrak{A}$ is of finite type, then there exists a map $A \to A^\natural$ of $\mathfrak{A}$ onto its center Z which has certain natural algebraic properties and reduces to the von Neumann-Murray relative trace when $\mathfrak{A}$ is a factor (i.e., when Z is one-dimensional). The present author announces some further results of this nature. Among them are the following. The classification of factors into types I, II, and III can be extended in a natural way to rings with arbitrary centers and the type of the commutator of a ring is the same as that of the ring itself. For rings $\mathfrak{A}$ of finite type the invariant C of Murray and von Neumann generalizes from a number to a numerical function on the set X of maximal ideals of Z. If $C \geq 1$ then there exists $x \epsilon H$ with $x \neq 0$ such that $(Ax, x) \geq (A^\natural x, x)$ for all A in $\mathfrak{A}$. Approximate finiteness generalizes naturally and two approximately finite rings of finite type are isomorphic if and only if their centers are isomorphic.

G. W. Mackey (Cambridge, Mass.).

✶**Schatten, Robert. A Theory of Cross-Spaces.** Annals of Mathematics Studies, no. 26. Princeton University Press, Princeton, N. J., 1950. vii+153 pp. $2.50.

This volume contains a synthesis of the author's work on cross spaces [Trans. Amer. Math. Soc. **53**, 195–217 (1943); **54**, 498–506 (1943); Ann. of Math. (2) **47**, 73–84 (1946); **48**, 321–325 (1947); these Rev. **4**, 161; **5**, 99; **7**, 455; **10**, 128] and of his joint work with von Neumann [ibid. (2) **47**, 608–630 (1946); **49**, 557–582 (1948); these Rev. **8**, 31; **10**, 256]. The result has the benefit of a uniform presentation and certain extensions are clarified.

The initial operation in forming a cross-space of two linear spaces B_1, B_2 is the group theoretic process of forming the formal expressions $\sum_{i=1}^n f_i \otimes g_i$ and establishing equivalences for these based on the distributivity of $\otimes$. The resulting equivalence classes form a linear set corresponding to those transformations from B_2^* to B_1 with a finite dimensional range. The introduction of a norm α into this set and the completion of the resulting space yields various classes of transformations which constitute normed spaces, and each transformation of the resulting class can be approximated in the α norm by transformations with finite dimensional ranges. To study these classes, the norms α can be restricted to "uniform cross-norms." A norm α is a cross-norm if $\alpha(f \otimes g) = |f| \cdot |g|$. A cross-norm is uniform if $\alpha(\sum Tf_i \otimes Sg_i) \leq \|\|T\|\| \cdot \|\|S\|\| \alpha(\sum f_i \otimes g_i)$ for every bounded transformation T on B_1 and S on B_2. The bound of the transformation corresponding to $\sum f_i \otimes g_i$ is a uniform cross-norm, $\lambda(\sum f_i \otimes g_i)$. To each norm α there corresponds a "conjugate" norm α' on the expressions from the conjugate spaces $B_1^* \otimes B_2^*$. The conjugate α' of a cross-norm α is a cross-norm if and only if $\alpha \geq \lambda$. Also, λ is local, i.e., $\lambda(\sum f_i \otimes g_i)$ depends only on the geometry of the linear manifolds determined by the f_i's and g_i's, respectively. The conjugate $y(\sum f_i \otimes g_i) = \lambda'$ of λ can be defined as lim inf $\sum \|f_i'\| \cdot \|g_i'\|$ of the expressions $\sum f_i' \otimes g_i'$ equivalent to $\sum f_i \otimes g_i$; y is a uniform cross-norm and any other cross-norm α is $\leq y$. However, y is not "local" in general, and if y is local, the space must be unitary.

Thus the machinery has been set up for studying the normed spaces $B_1 \otimes_\alpha B_2$ and $B_1^* \otimes_{\alpha'} B_2^*$ determined by the conjugate norm α', in a form equivalent to the study of classes of transformations in which a norm has been introduced with certain desirable properties. The space $B_1^* \otimes_{\alpha'} B_2^*$ is equivalent to a subspace of $(B_1 \otimes_\alpha B_2)^*$, and $(B_1 \otimes_\alpha B_2)^*$ is also proved to be isomorphic to a class of transformations, those of "finite α norm," from B_1 to B_2^*. For $\alpha = y$, this class contains all bounded transformations. Also, $B_1^* \otimes_{\alpha'} B_2^*$ corresponds to that subclass of $(B_1 \otimes_\alpha B_2)^*$ which can be approximated in the α norm by transformations with finite dimensional ranges, and for $\alpha \geq \lambda$ the transformations of $B_1^* \otimes_{\alpha'} B_2^*$ are completely continuous. For a uniform cross-norm, the operators A of finite α norm have the property that with A, XAY is in the set, X and Y bounded, and the author designates a set of transformations of finite α norm, with this property, as an "ideal" when α is uniform. Thus for a uniform α, $(B_1 \otimes_\alpha B_2)^*$ is an ideal of operators and $B_1^* \otimes_{\alpha'} B_2^*$ is an ideal and in general a proper sub-ideal.

When the spaces B_1 and B_2 are specialized to be Hilbert spaces, uniformity for a cross-norm becomes equivalent to unitary invariance and implies $\alpha'' = \alpha$. In addition to y and λ, the norm for the Schmidt class σ is introduced. If A is a transformation in the Schmidt class, then $\sigma(A)^2 = \sum |(A\varphi_i, \psi_j)|^2$ is finite for any pair of complete orthonormal sets $\{\varphi_i\}$ and $\{\psi_j\}$ and independent of the

choice of the complete sets. The cross-space $\mathfrak{H} \otimes_\sigma \bar{\mathfrak{H}}$ is of course a Hilbert space with associate $\sigma' = \sigma$ and $(\mathfrak{H} \otimes_\sigma \bar{\mathfrak{H}})^* = \mathfrak{H} \otimes_{\sigma'} \bar{\mathfrak{H}} = \mathfrak{H} \otimes_\sigma \bar{\mathfrak{H}}$. The linear space of all completely continuous operators with the bound as norm is $\mathfrak{H} \otimes_\lambda \bar{\mathfrak{H}}$. The class of transformations in the form $A = BC$, where B and C are in the Schmidt class, is called a trace. This class can be normed by y and $(\mathfrak{H} \otimes_\lambda \bar{\mathfrak{H}})^* = \mathfrak{H} \otimes_y \bar{\mathfrak{H}}$. Notice that the more general result on y established above shows that $(\mathfrak{H} \otimes_\lambda \bar{\mathfrak{H}})^{**} = (\mathfrak{H} \otimes_y \bar{\mathfrak{H}})^*$ is the class of all bounded transformations. To each completely continuous transformation A there exist two orthonormal sets $\{\varphi_i\}$ and $\{\psi_i\}$ and a sequence of positive constants $\{a_i\}$ such that $Af = \sum a_i (f, \psi_i) \varphi_i$ and $\lim a_i = 0$. As a dyadic, A is expressed: $A = \sum a_i \varphi_i \otimes \psi_i$. Uniform invariance for a norm α is equivalent to dependence on the a_i's, and a sufficient condition is obtained for those α's associated with classes of completely continuous transformations, so that $(\mathfrak{H} \otimes_\alpha \bar{\mathfrak{H}})^* = \mathfrak{H} \otimes_{\alpha'} \bar{\mathfrak{H}}$.

In appendix I, the associate norm α' is discussed for reflexive spaces. Thus $\alpha' = \alpha'''$ and the property $\alpha'' = \alpha$ is shown to be equivalent to the property $\alpha = \beta'$ and also to the property that $\alpha' = \beta'$ implies $\alpha \leq \beta$. Based on the author's joint work with Dunford, appendix II shows that even when $\alpha'' = \alpha$ and both B_1 and B_2 are reflexive, the cross-space $B_1 \otimes_\alpha B_2$ is not necessarily reflexive [Trans. Amer. Math. Soc. 59, 430–436 (1946); these Rev. 7, 455]. Appendix III shows that for any two Banach spaces B_1 and B_2, there exists a self-associate cross-norm which specializes to σ when B_1 and B_2 are unitary. *F. J. Murray.*

Hartman, Philip, and Wintner, Aurel. On the spectra of Toeplitz's matrices. Amer. J. Math. 72, 359–366 (1950).

Let f_n, $n = 0, \pm 1, \cdots$, be a sequence of complex numbers with $f_{-n} = \bar{f}_n$ and $\sum |f_n|^2 < \infty$. The Laurent, Toeplitz, and Hankel matrices defined by it are $L = (f_{n-m})$ with $n, m = 0, \pm 1, \cdots$, $T = (f_{n-m})$ with $n, m = 0, 1, \cdots$ and $H = (f_{n+m+1})$ with $n, m = 0, 1, \cdots$. The spectral analysis of L, or more generally of the analogous operator when the additive group of the integers is replaced by a locally compact Abelian group, is well known. In this article some facts concerning the spectral analysis, boundedness, and complete continuity of T and H and their relationship to the corresponding facts for L are presented. *L. Nachbin* (Rio de Janeiro).

Krasnosel'skiĭ, M. A. Convergence of Galerkin's method for nonlinear equations. Doklady Akad. Nauk SSSR (N.S.) 73, 1121–1124 (1950). (Russian)

Let $L_1 \subset L_2 \subset \cdots$ be a sequence of finite-dimensional vector spaces whose union is the Banach space E, and let $P_1, P_2, \cdots$ be projection operators of uniformly bounded norms such that $P_n E = L_n$. Then the equations $\varphi = P_n A \varphi$ and their solutions φ_n ($n = 1, 2, \cdots$) are said to approximate the equation $\varphi = A\varphi$ and its solution φ_0. If the L_n are spanned by the first n elements of a basis of E, then the approximations are called Galerkin approximations by the author. The following theorems are stated without proof. If A is a completely continuous (nonlinear) operator in E and if the equation $\varphi = A\varphi$ has a unique (or at least unique in some sphere of E) solution whose topological index is different from zero, then the approximating equations, starting with a certain n, have solutions φ_n and these converge to φ_0. If the operator A has, at the point φ_0, a Fréchet differential B for which 1 is not a characteristic value, then the approximating solutions φ_n converge to φ_0 essentially like the "Fourier series" $P_1\varphi_0 + (P_2 - P_1)\varphi_0 + (P_3 - P_2)\varphi_0 + \cdots$,

viz., $\|\varphi_n - \varphi_0\| \leq (1 + \epsilon_n)\|P_n\varphi_0 - \varphi_0\|$, where $\lim_{n \to \infty} \epsilon_n = 0$. If E is a real Hilbert space and A is a completely continuous linear operator which has the simple characteristic value $\lambda_0 > 0$ and corresponding characteristic element φ_0 for which $\|\varphi_0\| = \lambda_0$, then φ_0 may be considered as a solution of the nonlinear equation $\varphi = \|\varphi\| A\varphi$, and the approximating equations become $\varphi = P_n \|\varphi\| A\varphi$. The above results then apply to the convergence of the approximating solutions φ_n to φ_0 and of the numbers $\|\varphi_n\|$ to λ_0. *M. Golomb.*

Krasnosel'skiĭ, M. A. On a topological method in the problem of characteristic functions of nonlinear operators. Doklady Akad. Nauk SSSR (N.S.) 74, 5–7 (1950). (Russian)

Let S be the boundary of a bounded open set in a real Banach space E containing the zero point θ, and let F be a completely continuous mapping of S into E. Assume $(F - I)\varphi \neq \theta$, $\varphi \epsilon S$. Following Leray and Schauder [Ann. Sci. École Norm. Sup. (3) 51, 45–78 (1934)] and E. H. Rothe [Compositio Math. 5, 177–197 (1937)] the topological index of the vector field $(F - I)\varphi$ on S is defined. The following theorem is stated: If $(F_1 - I)\varphi$, $(F_2 - I)\varphi$ are two such vector fields of different topological index, then the equation $\varphi = \mu F_1\varphi + (1 - \mu)F_2\varphi$ has at least one solution on S for some μ, $0 < \mu < 1$. This theorem leads to another one concerning characteristic (eigen) elements of a completely continuous (nonlinear) operator A for which $A\theta = \theta$. A set of characteristic elements φ of A belonging to characteristic values lying in some interval about μ, for which $\varphi \to \theta$ as the corresponding characteristic values tend to μ and which has a nonvoid intersection with the boundary of every sufficiently small open set containing θ, is said to be a branch of characteristic elements terminating in θ with the characteristic value μ. The theorem states that if the operator A has a Fréchet differential B at the point θ, then for every characteristic value λ_k of B of odd multiplicity there exists a branch of characteristic elements of A terminating in θ with the characteristic value λ_k. Applications to the theory of nonlinear integral equations are pointed out. No proofs are given. *M. Golomb* (Lafayette, Ind.).

Krasnosel'skiĭ, M. A. Characteristic functions of nonlinear operators which are asymptotically near to linear ones. Doklady Akad. Nauk SSSR (N.S.) 74, 177–179 (1950). (Russian)

As a counterpart to the branches of characteristic elements terminating in θ [see the preceding review], the author considers branches of characteristic elements tending to ∞ with the asymptotic characteristic value λ. These are sets of characteristic elements φ of the operator A belonging to characteristic values lying in some interval about λ, for which $\|\varphi\| \to \infty$ as the corresponding characteristic values tend to λ and which have a nonvoid intersection with the boundary of every open set that contains a sufficiently large sphere. The following theorem is stated: If the completely continuous (nonlinear) operator A is asymptotically close to the linear operator B, then for every characteristic value λ_k of B of odd multiplicity there exists a branch of characteristic elements of A with the asymptotic characteristic value λ_k. The statement "A is asymptotically close to B" means $\lim_{\rho \to \infty} \sup_{\|\varphi\| = \rho} \|A\varphi - B\varphi\|/\rho = 0$. Applications to characteristic value problems for systems of nonlinear integral equations are indicated. *M. Golomb.*

Morse, Marston, and Transue, William. Norms of distribution functions associated with bilinear functionals. Contributions to Fourier Analysis, pp. 104–144. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. $3.00.

This paper is one of a series of studies in the theory of functionals bilinear on the Cartesian product $A \times B$ of two pseudo-normed vector spaces [Canadian J. Math. 1, 153–165 (1949); Proc. Nat. Acad. Sci. U. S. A. 35, 136–143, 395–399 (1949); Ann. of Math. (2) 50, 777–815 (1949); Rivista Mat. Univ. Parma 1, 3–18 (1950); these Rev. 10, 601, 612; 11, 19, 185, 512]. In view of the number and complexity of basic definitions, reference must be made to these previous reviews. The main object of the present paper is to compare two total variations, namely, the variation based on ideas of Fréchet and used by the authors, and the variation proposed by Vitali and used in alternative theories. This comparison establishes the superiority of the first one of these variations. The results represent far-reaching improvements over previous literature.

T. Radó (Columbus, Ohio).

Fullerton, R. E. On a semi-group of subsets of a linear space. Proc. Amer. Math. Soc. 1, 440–442 (1950).

The author proves essentially the following result. Let X be a linear space over the reals. Let S be a subset of X such that if x, y, $\alpha_n x$, $\beta_n y \varepsilon S$ with $\alpha_n + \beta_n = 1$ and $\alpha_n \to \alpha$, $\beta_n \to \beta$, then $\alpha x + \beta y \varepsilon S$. Suppose S contains an extreme point, and the family of all translates of S in X forms a semi-group under intersection. Then S is a convex cone. As an application, a slight generalization of Clarkson's characterization of the Banach space of continuous real functions over a bicompact Hausdorff space follows [Ann. of Math. (2) 48, 845–850 (1947); these Rev. 9, 192]. *S. B. Myers.*

Jerison, Meyer. The space of bounded maps into a Banach space. Ann. of Math. (2) 52, 309–327 (1950).

Soient B un espace de Banach, X un espace topologique, B^X l'espace de Banach des applications continues et bornées de X dans B. Généralisant les résultats de Eilenberg et Myers relatifs au cas où B est l'ensemble des nombres réels, l'auteur se propose de caractériser la topologie de X au moyen de propriétés de la norme dans B^X; il faut pour cela faire des hypothèses restrictives sur B, comme le montre un contre-exemple. On aboutit à une telle caractérisation en supposant que la boule unité de B est strictement convexe; la méthode consiste à considérer les parties convexes maximales de la sphère unité dans B^X et les cônes engendrés par ces parties: lorsque X est compact, on montre en effet qu'un tel ensemble correspond biunivoquement à un couple (x, L) formé d'un point de X et d'une partie convexe maximale de la sphère unité de B. L'auteur donne ensuite un système de 5 conditions nécessaires et suffisantes pour qu'un espace de Banach E soit de la forme B^X, où X est un espace compact et B un espace de Banach strictement convexe convenables. Il remarque enfin que si X est un espace complètement régulier, Y sa compactification de Stone-Čech, il n'est pas toujours possible de prolonger à Y une application continue de X dans un espace de Banach B de dimension infinie. [Le rapporteur signale que la définition d'un isomorphisme d'un espace de Banach B sur un autre B', rappelée dans la définition 2.1, n'est correcte que si on remplace le mot "into" par "onto".] *J. Dieudonné* (Baltimore, Md.).

Segal, I. E. The class of functions which are absolutely convergent Fourier transforms. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 157–161 (1950).

Let G be a locally compact Abelian group, G^* its character group, and C the Banach space of continuous complex-valued functions on G with the uniform norm. The author shows that the subset of C consisting of all absolutely convergent Fourier transforms (Fourier transforms of functions in $L_1(G^*)$) is of the first category in C if G is infinite, and is the whole of C if G is finite. *L. H. Loomis.*

Šnol', I. È. The structure of ideals in rings R_α. Mat. Sbornik N.S. 27(69), 143–146 (1950). (Russian)

The rings R_α considered here are normed rings of continuous functions contained between the ring C of all continuous functions on $[0, 1]$ and the ring D_1 of all functions with continuous first derivatives on $[0, 1]$. These rings were studied by Šilov [Mat. Sbornik N.S. 26(68), 291–310 (1950); these Rev. 11, 602]. In the present paper it is proved that every closed ideal in a ring R_α is the intersection of primary ideals. This result was obtained by Stone [Trans. Amer. Math. Soc. 41, 375–481 (1937)] for the ring C and by Šilov for the ring D_1 [cf. Trav. Inst. Math. Stekloff 21 (1947), § 5, chapter I and reference cited there; these Rev. 9, 596]. *C. E. Rickart* (New Haven, Conn.).

Berezanskiĭ, Yu. M., and Kreĭn, S. G. Continuous algebras. Doklady Akad. Nauk SSSR (N.S.) 72, 5–8 (1950). (Russian)

Par une "algèbre continue," les auteurs entendent ce qui suit: c'est une algèbre associative et commutative sur le corps complexe, done les éléments peuvent être identifiés aux fonctions complexes continues définies sur un espace topologique compact Q (qui joue le rôle d'une "base" de l'algèbre), et ce de telle sorte que, si l'on note $f*g$ la multiplication dans l'algèbre, on ait une formule du type $f*g(x) = \iint_{Q \times Q} f(y)g(z)dC_x(y, z)$; dans cette formule, C_x désigne une mesure de Radon sur $Q \times Q$, qui est positive et dépend continûment du point $x \varepsilon Q$. En fait, la définition donnée par les auteurs n'est pas aussi simple que ce qui précède, car ils parlent uniquement de fonctions d'ensemble; mais l'équivalence des deux définitions se voit immédiatement. Étant donnée une algèbre continue de base Q, les auteurs appellent mesure multiplicative toute mesure positive m sur Q telle que l'on ait $m(f*g) = m(f)m(g)$ pour f, g continues sur Q (ici encore la définition des auteurs est en apparence différente); par une application ingénieuse d'un théorème de M. Kreĭn sur les opérateurs conservant un cône, les auteurs prouvent qu'il existe toujours de telles mesures (non nulles!) (le fait que les mesures C_x sont positives semble essentiel pour la démonstration); en outre, deux telles mesures sont toujours absolument continues l'une par rapport à l'autre, au moins si, pour tout $x \varepsilon Q$ il existe f, g avec $f*g(x) \neq 0$.

Soit dx une mesure multiplicative; la multiplication $f*g$ est alors prolongeable par continuité à l'espace L^1 construit sur dx, d'où une algèbre normée complète et commutative. Une fonction $\chi(x)$, mesurable et bornée pour dx, est dite un caractère si la formule $f \to \int f(x)\chi(x)dx$ définit un homomorphisme de l'algèbre L^1 sur le corps des nombres complexes; ces caractères sont évidemment en correspondance avec les idéaux maximaux de l'algèbre déduite de L^1 par adjonction d'un élément unité; il peut du reste arriver qu'il n'existe pas de caractère non trivial (autrement dit, l'algèbre

L^1 peut être identique à son radical). Les auteurs donnent finalement trois exemples d'algèbres continues; le premier est évidemment celui des groupes compacts abéliens; le second est relatif à $Q=(-1, +1)$ et a pour caractères les polynomes de Legendre; le troisième, analogue, conduit aux polynomes de Tchebicheff. Il est à remarquer que, dans ces deux derniers exemples, les auteurs définissent directement $f*g$, autrement dit, utilisent la définition que nous avons donnée au début, au lieu de la leur. Bien entendu, et comme les auteurs le font eux-mêmes observer, les notions introduites ici sont en relation étroite avec la théorie des "systèmes de translations généralisées" de Levitan. Par ailleurs, il serait souhaitable de généraliser leur théorie au cas des espaces localement compacts, et de se libérer de l'hypothèse que les C_s sont positives; on pourrait alors inclure dans la théorie les "fonctions sphériques" de Gelfand, par exemple. *R. Godement* (Nancy).

Berezanskiĭ, Yu. M., and Kreĭn, S. G. Some classes of continuous algebras. Doklady Akad. Nauk SSSR (N.S.) 72, 237–240 (1950). (Russian)

Une algèbre continue [cf. l'analyse ci-dessus] est dite normale s'il existe sur Q un automorphisme involutif $x \to x^*$ tel que la formule $f^*(x) = \overline{f(x^*)}$ définisse un antiautomorphisme involutif pour la multiplication $f*g$. Toute mesure multiplicative est alors invariante par $x \to x^*$. C'est évidemment le cas des algèbres de groupes $(x^* = x^{-1})$; et aussi des deux derniers exemples de l'analyse précédente (dans ce cas, $x^* = x$). Dans une algèbre continue normale, le produit de composition de deux fonctions sommables (pour une mesure multiplicative donnée) est une fonction continue; on en déduit que les caractères d'une telle algèbre sont eux-mêmes des fonctions continues; de plus ils vérifient $\chi(x^*) = \overline{\chi(x)}$, sont en infinité dénombrable (les auteur supposent Q séparable) et deux à deux orthogonaux. Il s'ensuit qu'une telle algèbre possède seulement une mesure multiplicative. Les éléments f du radical de L^1 (qui peut ne pas être nul) sont caractérisés comme suit: quelles que soient $g, h \varepsilon L^1$ on a $f*g*h = 0$ (la démonstration utilise essentiellement la représentation unitaire évidente de L^1 dans L^2). On dit qu'un point $o \varepsilon Q$ est un "pôle" de l'algèbre si, quels que soient les ensembles boréliens $A, B \subset Q$, on a $C_o(A \times B) = m(A^* \cap B)$ (C_o est la mesure "structurale" du point o; m est la mesure multiplicative). S'il existe un pôle, l'algèbre L^1 est semi-simple, les caractères forment une base orthonormale de L^2, et si les "coefficients de Fourier" d'une fonction bornée sont tous positifs, la série de Fourier correspondante est absolument et uniformément convergente sur Q, phénomène bien connu dans le cas des groupes, et qui s'applique aussi, par exemple, aux développements en séries de polynomes de Legendre. *R. Godement* (Nancy).

Berezanskiĭ, Yu. M. On the center of the group ring of a compact group. Doklady Akad. Nauk SSSR (N.S.) 72, 825–828 (1950). (Russian)

Let G be a separable compact group and let Q be the set of all classes of conjugate elements in G. Then Q is compact in a natural topology and, via the natural mapping of G on Q, Haar measure in G induces a measure m in Q. This same mapping of G on Q induces a one-to-one linear norm preserving map of the center of the group ring of G onto $\mathfrak{L}'(Q, m)$. In this way $\mathfrak{L}'(Q, m)$ is made into a commutative normed ring. It is shown that $\mathfrak{L}'(Q, m)$ is actually a commutative continuous algebra in the sense of earlier work of the author and S. Kreĭn [see the two preceding reviews].

The theory of commutative continuous algebras is then applied to show that there is a natural one-to-one correspondence between the "characters" of $\mathfrak{L}'(Q, m)$ and the characters of the irreducible representations of G and hence to prove anew that G has "sufficiently" many irreducible unitary representations. *G. W. Mackey.*

Mikusiński, Jan G.-. L'anneau algébrique et ses applications dans l'analyse fonctionnelle. II. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 3, 1–84 (1949). (French. Polish summary)

Continuation of a previous paper [same Ann. Sect. A. 2, 1–48 (1947); these Rev. 10, 259, 856]. This paper contains with more details the results of another paper [see the following review], but most of these details could have been avoided by using the classical results of algebra, topology, or functional analysis. *L. Schwartz* (Nancy).

Mikusiński, Jan G.-. Sur les fondements du calcul opératoire. Studia Math. 11, 41–70 (1949).

Let $\mathfrak{A}$ be the commutative algebra of the complex continuous functions $f(x)$ over the half-straight-line $x \geq 0$, the multiplication being defined by the convolution

$$(f, g) \to f \cdot g = \int_0^x f(x-t)g(t)dt.$$

The topology is defined by uniform convergence over every compact set. This algebra has no zero divisors [Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937, p. 327; Crum, Quart. J. Math., Oxford Ser. (1) 12, 108–111 (1941); these Rev. 3, 39; Dufresnoy, C. R. Acad. Sci. Paris 225, 857–859 (1947); these Rev. 9, 237]. Thus it is possible to introduce the field of quotients $\mathfrak{Q}$ of $\mathfrak{A}$, the elements of which are abstract entities, giving a generalization of the notion of function. Dirac's measure δ is the unity of $\mathfrak{Q}$. Let Y be Heaviside's function, $Y(x) = 1$ for $x \geq 0$. Then $p = (Y)^{-1}$ is an element of $\mathfrak{Q}$; every $T \varepsilon \mathfrak{Q}$ is differentiable with $T' = p \cdot T$. This generalization, and this notion of differentiation are very similar to those introduced by the reviewer [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 57–74 (1946); these Rev. 8, 264]. However, $\mathfrak{Q}$ and the space $(\mathfrak{D}')$ of distributions are quite different; some elements of $\mathfrak{Q}$ are not in $(\mathfrak{D}')$ (example: $(\alpha)^{-1}$, if $\alpha \varepsilon (\mathfrak{D})$), and some elements of $(\mathfrak{D}')$ are not in $\mathfrak{Q}$ (but all the distributions in $(\mathfrak{D}')$ having their support in the half-straight-line $x \geq 0$ are in $\mathfrak{Q}$). No notion of support exists here. The ordinary multiplicative product does not seem to exist. No generalization is possible for the whole real axis.

A pseudotopology is introduced in $\mathfrak{Q}$ in the following way. An application $\lambda \to X(\lambda)$ of a topological space Λ into $\mathfrak{Q}$ will be continuous (respectively, differentiable, ...) if there is a fixed element q of $\mathfrak{A}$ such that $\lambda \to q \cdot X(\lambda)$ of Λ into $\mathfrak{A}$ is continuous (respectively, differentiable, ...). It is proved that any differential equation of the form $a_n X^{(n)}(\lambda) + a_{n-1} X^{(n-1)}(\lambda) + \cdots + a_0 X(\lambda) = B(\lambda)$ has at most one solution for initial Cauchy values (the proof is given for $n=1$, partially for $n=2$ in another paper [Ann. Soc. Polon. Math. 22, 157–160 (1949); these Rev. 12, 8], and for any n by Drobot and Mikusiński [Studia Math. 11, 38–40 (1949); these Rev. 12, 9]). From this it follows that the exponential $\exp(\lambda T)$, when it is defined, is uniquely determined by the differential equation $X'(\lambda) = T \cdot X(\lambda)$, and the initial condition $X(0) = \delta$. If this exponential exists in the interval $\lambda_1 \leq \lambda \leq \lambda_2$, it exists for any λ. In particular, $\exp(-p\lambda)$ exists; it is the mass $+1$ on the point with

abscissa λ. By integration the following formula (given by Ryll-Narzewsky) is obtained: $\int_0^\infty \exp(-p\lambda)f(\lambda)d\lambda = f$, for $f \varepsilon \bar{a}$. This gives a new interpretation of the Laplace transform. Moreover, $\exp(-ip^i\lambda)$ has a meaning. These exponentials have significance in the wave equation $X''(\lambda) - p^2 X(\lambda) = 0$, and the heat equation $X''(\lambda) - pX(\lambda) = 0$. This theory gives satisfying formulae for symbolic calculations. Some applications are given (uniqueness theorems for partial differential equations, solution of boundary value problems, . . .).

$\qquad\qquad\qquad$ *L. Schwartz* (Nancy).

Mikusiński, Jan G.-.   Une nouvelle justification du calcul de Heaviside. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) Sez. I. **2**, 113–121 (1950).

Summary of a longer paper [see the preceding review]. The generalized functions are introduced here by a topological method, instead of the algebraic one used above.

$\qquad\qquad\qquad$ *L. Schwartz* (Nancy).

Yosida, Kôsaku.   An operator-theoretical treatment of temporally homogeneous Markoff process. J. Math. Soc. Japan **1**, 244–253 (1949).

Previous results of the author [same vol., 15–21 (1948); these Rev. **10**, 462] concerning the differential quotient operator (d.q.o.) A of a strongly continuous one-parameter semi-group $\{U_t | 0 \leq t < \infty\}$ of bounded linear operators U_t on a Banach space E are applied to the case of a temporally homogeneous Markoff process, i.e., to the case when E is an (L)-space and U_t is a transition operator (i.e., a positive linear operator which preserves the norm of positive elements). $Ax = \text{weak } \lim_{t \to 0} t^{-1}(U_t - I)x$ exists on a dense linear subspace D of E, and A is characterised as a closed linear operator with the following two properties: (i) $I_n = n(I - n^{-1}A)^{-1}$ exists and is a transition operator; (ii) $AI_n = n(I - I_n)$ and $\lim_{n \to \infty} AI_n x = Ax$ for all $x \varepsilon D$. The author takes the case when $E = L(-\infty, \infty)$ and shows that (iii) $Ax(s) = x'(s)$ if U_t is a translation: $U_t x(s) = x(s + t)$, (iv) $Ax(s) = \frac{1}{2}x''(s)$ if U_t is a Gaussian convolution:

$$U_t x(s) = (\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-(u-s)^2/t)x(u)du,$$

(v) $Ax(s) = \lambda(x(s-u) - x(s))$ if U_t is a Poisson convolution: $U_t x(s) = \exp(-\lambda t)\sum_{k=0}^{\infty}[(\lambda t)^k/k!]x(s-ku)$. The author also obtains an expression for A when U_t is a convolution with a general infinitely divisible law. These results were previously obtained independently by Hille [Functional Analysis and Semi-groups, Amer. Math. Soc. Colloquium Publ., v. 31, Amer. Math. Soc., New York, 1948; these Rev. **9**, 594]. The author further applies his method to the problem of integration of Fokker-Planck's equation and shows that the operator A defined by

$$Ax(s) = (a(s)x(s))' + (b(s)x(s))''$$
$$\equiv \delta(s)x(s) + \gamma(s)y'(s) + \frac{1}{2}\beta(s)y''(s)$$

satisfies the conditions (i), (ii) above and hence is a d.q.o. of a temporally homogeneous Markoff process if $\delta(s)$, $\gamma(s)$, $(\beta(s))^{-1}$, $\beta'(s)$ are all bounded, continuous, $\beta(s) \geq 0$, and

$$\int_0^\infty [\beta(s)]^{-\frac{1}{2}}ds = \infty, \quad \int_{-\infty}^0 [\beta(s)]^{-\frac{1}{2}}ds = \infty.$$

$\qquad\qquad\qquad$ *S. Kakutani* (New Haven, Conn.).

Anzai, Hirotada.   Random ergodic theorem with finite possible states. Osaka Math. J. **2**, 43–49 (1950).

For every integer k, let X_k be the measure space consisting of the first p positive integers each carrying measure

$1/p$ (where p is a fixed positive integer), and let X be the Cartesian product of all X_k. Let Y be a measure space consisting of q points $y_1, \cdots, y_q$, each carrying measure $1/q$ (where q is a fixed positive integer). Let $T = \{T_1, \cdots, T_p\}$ be a family of p permutations of Y. Given a point y in Y, select a random element $x = \{x_k\}$ from X, and form the sequence $y, T_{x_1}y, T_{x_2}T_{x_1}y, \cdots$. If $t_{ij}{}^{(n)}$ is the conditional probability that the nth term of this sequence is y_i, given that $y = y_j$, and if $\lim_{n \to \infty} t_{ij}{}^{(n)} = 1/q$ for all i and j, the family T is called strongly mixing. If s is the coordinate-shift transformation on X, and if $S(x, y) = (s(x), T_{x_1}y)$, then S is a measure preserving transformation on the Cartesian product of X and Y. The author's two results are that (theorem 1) S is strongly mixing if and only if the family T is strongly mixing, and (theorem 2) S is ergodic if and only if the family T is ergodic (in the sense that Y contains no nontrivial set invariant under all the permutations of T).

$\qquad\qquad\qquad$ *P. R. Halmos* (Chicago, Ill.).

Anzai, Hirotada.   Mixing up property of Brownian motion. Osaka Math. J. **2**, 51–58 (1950).

The author's main result is that if (1) X is the space of a one-dimensional Brownian motion, (2) φ_t is the translation flow on X, (3) $\alpha(t, x) = x(t) - x(0)$, (4) ψ_t is a measurable ergodic flow on a measure space Y, and (5) T_t is the flow on $X \times Y$ defined by $T_t(x, y) = (\varphi_t x, \psi_{\alpha(t, x)}y)$, then T_t is strongly mixing; as a corollary he deduces a continuous random ergodic theorem. The conclusion of the main theorem (i.e., the mixing property of the "skew product flow" T_t) is contrasted with the situation for ordinary product flows (where $\alpha(t, x) = t$); in that case the ergodicity of ψ_t is not sufficient to imply the mixing property of T_t. The crucial property of the function α (in addition to certain measurability conditions) is the fact that it satisfies the functional equation $\alpha(s, x) + \alpha(t, \varphi_s x) = \alpha(s+t, x)$. The author shows that for certain flows φ_t on certain spaces X no such result as holds for Brownian motions is possible. More precisely, he shows that if X is a circle of unit length, if φ_t is the rotation flow on X, and if a function α satisfies the measurability conditions and the functional equation, then $\alpha(t, x) - ct = \beta(\varphi_t x) - \beta(x)$ for a suitable measurable function β and constant c; it follows that every skew product flow whose first factor is this φ_t is equivalent to an ordinary product flow. $\qquad\qquad$ *P. R. Halmos* (Chicago, Ill.).

Theory of Probability

Usai, Giuseppe.   Valor medio della potenza di una variabile casuale nel problema delle prove ripetute. Atti Accad. Gioenia Catania (6) **6** (1943/49), no. 8, 7 pp. (1950).

Let X be the random variable with the binomial probability distribution $P(X) = \binom{n}{x}p^X(1-p)^{n-X}$, $X = 0, 1, \cdots, n$. It is proved that $E(X^r) = \sum_{s=1}^r a_{r,s}p^s n!/(n-s)!$, where $a_{r,s}$ are elements of an "arithmetical triangle" defined by $a_{r,1} = a_{r,r} = 1$ and $a_{r,s} = sa_{r-1,s} + a_{r-1,s-1}$ for $s = 2, 3, \cdots, r-1$, and $r = 1, 2, 3, \cdots$. Some other properties of the numbers $a_{r,s}$ are also discussed.   *Z. W. Birnbaum* (Seattle, Wash.).

Amato, Vittorio.   Sui limiti di applicabilità della formula di Poisson. Statistica, Milano **10**, 149–152 (1950).

The author derives a condition under which the probability of x successes in n trials of the Bernoulli probability function $P_x = {}_nC_x p^x q^{n-x}$ may be approximated by the Poisson

probability function $e^{-m}m^x/x!$. The condition states that $|1-\alpha^2|p<\frac{1}{2}$, where $\alpha=(x-m)/(npq)^{\frac{1}{2}}$. *L. A. Aroian.*

Sapogov, N. A. **On a property of the Gaussian distribution law.** Doklady Akad. Nauk SSSR (N.S.) **73**, 461–462 (1950). (Russian)

A proof of the following theorem is sketched. Suppose that the distribution function F of the sum of two mutually independent random variables satisfies the inequality $|F(x)-\Phi(x)|<\epsilon<1$ for all x, where Φ is the normal distribution function with mean 0 and variance 1. Let F_1 be the distribution function of either summand and suppose that $a_1=\int_{-N}^{N}xdF(x)$ and that $\sigma_1^2=\int_{-N}^{N}x^2dF_1(x)-a_1^2>0$, where $N=(\log 1/\epsilon)^{\frac{1}{2}}$. Then there is an absolute constant C for which $|F_1(x)-\Phi((x-a_1)/\sigma_1)|<C\sigma_1^{-8/7}(\log 1/\epsilon)^{-2/7}$ for all x. This theorem generalizes Cramér's theorem [Math. Z. **41**, 405–414 (1936)] that, if the sum of two mutually independent random variables is normally distributed, each summand also has this property. *J. L. Doob.*

Bhabha, H. J. **On the stochastic theory of continuous parametric systems and its application to electron cascades.** Proc. Roy. Soc. London. Ser. A. **202**, 301–322 (1950).

The author supposes that there is a system of particles on a line, subject to the following stochastic scheme. The probability that there are k particles is given for each k. If there are k, their k-variate probability distribution is determined by a k-dimensional symmetric density function. It is shown how to calculate the mean of various functional averages over the particles; for example, the expected number and the expectation of the square of the number of particles in a given interval are calculated. The theory is then applied to electron cascades, in which at a given time the particles are characterized by their energy values, so that the energy value takes the place of position in the above. *J. L. Doob* (Urbana, Ill.).

Itô, Kiyosi. **On a stochastic integral equation.** Proc. Japan Acad. **22**, nos. 1–4, 32–35 (1946).

For each $t\geq 0$ let $g(t)$ be a random variable, and suppose that the $g(t)$ stochastic process is the Brownian motion process. The author proves that there is an $x(t)$ process, almost all of whose sample functions are continuous, satisfying the equation

$$x(t)=c+\int_0^t a[s,x(s)]ds+\int_0^t b[s,x(s)]dg(s),\qquad 0\leq t\leq 1.$$

Here c is a constant, $a(t,x)$, $b(t,x)$ are continuous in t, x, and satisfy Lipschitz conditions in x, uniformly in t. The problem is closely related to the study of Markov processes whose transition probabilities are subject to the Fokker-Planck equation, and the solution gives new insight into the structure of such processes. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. **Sur l'aire comprise entre un arc de la courbe du mouvement brownien plan et sa corde.** C. R. Acad. Sci. Paris **230**, 432–434, 689 (1950).

Let $\{(X(t), Y(t))|0\leq t<\infty\}$, $(X(0)=Y(0)=0)$, be a Brownian motion on the plane. Let S be the "stochastic area" of a "domain" bounded by the curve $\{(X(t), Y(t))|0\leq t\leq 2\pi\}$ and the chord connecting its two end points $(0, 0)$, $(X(2\pi),$ $Y(2\pi))$ as defined by the author [Amer. J. Math. **62**, 487–550 (1940); these Rev. **2**, 107]. The author proves that

the characteristic function of the law of S is given by $(\cosh(\pi z))^{-1}$. The proof is based on the fact that if

$$X(t)+iY(t)=\frac{\xi_0+i\eta_0}{(2\pi)^{\frac{1}{2}}}t$$
$$+\sum_{n=1}^{\infty}\frac{(\xi_n+i\eta_n)(\cos nt-1)+(\xi_n'+i\eta_n')\sin nt}{n\pi^{\frac{1}{2}}},$$

is a representation of the Brownian motion in the sense of Paley-Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, Amer. Math. Soc., New York, 1934], where ξ_n, η_n, ξ_n', η_n' are mutually independent real-valued random variables with the same normalized Gaussian distribution with the mean value 0 and the variance 1, then S is given by

$$S=\sum_{n=1}^{\infty}\frac{\xi_n(\eta_n'-2^{\frac{1}{2}}\eta')-\eta_n(\xi_n'-2^{\frac{1}{2}}\xi')}{n}=\sum_{n=1}^{\infty}\frac{(\xi_n'')^2+(\eta_n'')^2}{2n+1},$$

where ξ_n'', η_n'' are linear orthogonal combinations of ξ_n, η_n, ξ_n', η_n'. From this follows that the law of S is infinitely divisible. Further, if U is any functional of $X(t)$ and $Y(t)$ which admits a representation of the form $U=\sum_{n=1}^{\infty}(\xi_n'')^2/2\lambda_n$, then the law of U is infinitely divisible and its characteristic function is given by $\prod_{n=1}^{\infty}(1-iz/\lambda_n)^{-\frac{1}{2}}$. *S. Kakutani.*

Arfwedson, Gerhard. **Some problems in the collective theory of risk.** Skand. Aktuarietidskr. **33**, 1–38 (1950).

The author discusses the collective theory of risk, taking into account the influence of the interest rate. This amounts to adding a linear term to the security (loading) factor. He discusses in detail the situation when the probability distribution of an individual claim has density e^{-x}. In this case he finds the probability that the risk reserve will remain positive during the interval $(0, x)$. Asymptotic expressions for this probability (which depends on x and on the initial reserve) are found in various cases of interest.

J. L. Doob (Urbana, Ill.).

Dwork, Bernard M. **Detection of a pulse superimposed on fluctuation noise.** Proc. I.R.E. **38**, 771–774 (1950).

The author's summary is as follows: "Given a known pulse superimposed on fluctuation noise having a known spectrum, we determine the frequency response of that linear device which would give the maximum value for the ratio between peak amplitude of the signal and the root-mean-square of the noise at the output. This result is applied to the case in which the fluctuation noise has a flat spectrum, and it is shown that in that case the optimal network is physically realizable if the pulse differs from zero for only a finite interval of time." *N. Levinson.*

Lee, Y. W., Cheatham, T. P., Jr., and Wiesner, J. B. **Application of correlation analysis to the detection of periodic signals in noise.** Proc. I.R.E. **38**, 1165–1171 (1950).

Mathematical Statistics

Lancaster, H. O. **The derivation and partition of χ^2 in certain discrete distributions.** Biometrika **36**, 117–129 (1949).

The general term in the multinomial distribution can be considered as the product of terms of binomial distributions.

To each of these there corresponds a value of χ^2 with one degree of freedom. These values of χ^2 are independent. This idea is extended to the case of $r \times s$ contingency tables, where the general term may be reduced to the product of terms of $(r-1)(s-1)$ 2×2 contingency tables. To each of these there is a value of χ^2 with one degree of freedom. These values of χ^2 are independent too. These results furnish a method of locating the specific places where the null hypothesis of independence in contingency tables breaks down (if it does break down). Alternative points of view using normal regression analysis and Helmert matrices are explored. Some examples are discussed. A slightly more sophisticated proof than that given is required because two uncorrelated normal deviates are not necessarily independent unless they have a joint normal distribution.

H. Chernoff (Urbana, Ill.).

Narain, R. D. Some results on discriminant functions. J. Indian Soc. Agric. Statistics **2**, 49–59 (1949).

The author's summary is as follows: "The general non-null distribution of the U-statistic has been obtained. It has been shown that the test of significance of a discriminant function fitted after eliminating certain variates which by themselves have no discrimination, as covariance variates, is in general more powerful than the test without using covariance variates, provided we consider sampling with respect to the covariance variates as well. The mth moment of the distribution of a discriminant function coefficient has been worked out and it has been shown that the distribution tends rapidly to normality." In equation 1.2, $w_1, w_2, \cdots, w_p$, of the first column of the determinant should be replaced by $a_{i1}, a_{i2}, \cdots, a_{ip}$. *H. Chernoff.*

Nanda, D. N. Distribution of the sum of roots of a determinantal equation under a certain condition. Ann. Math. Statistics **21**, 432–439 (1950).

Let $x = \|x_{ij}\|$, $x^* = \|x_{ij}^*\|$ be two p-variate sample matrices with n_1 and n_2 degrees of freedom and $S = \|xx'\|/n_1$, $S^* = \|x^*x^{*'}\|/n_2$, the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix. The joint distribution of the roots of the determinantal equation $|A - \theta(A+B)| = 0$, where $A = n_1 S$, $B = n_2 S^*$, $l = \min(p, n_1)$, $\mu = |p - n_1| + 1$, $v = n_2 - p_1 + 1$, $m = \frac{1}{2}\mu - 1$, $n = \frac{1}{2}v - 1$, has been investigated in two previous papers by the author [same Ann. **19**, 47–57, 340–350 (1948); these Rev. **9**, 453; **10**, 135]. In this paper he finds the distribution of the roots of the determinantal equation and the moments of the distribution under the condition that $m = 0$, for $l = 2, 3$, and 4. *L. A. Aroian* (Culver City, Calif.).

Bouzitat, J. Note sur un problème de sondage. O.N.E.R.A. Publ. no. 1, iv+57 pp. (6 plates) (1947).

"Le but de ce travail est d'étudier, à l'aide du calcul des probabilités, les renseignements que l'on peut obtenir sur la répartition inconnue a priori d'un nombre N d'objets entre plusieurs catégories, par l'examen d'un certain nombre n de ces objets choisis au hasard."

From the author's summary.

Sherman, B. A random variable related to the spacing of sample values. Ann. Math. Statistics **21**, 339–361 (1950).

Let $x_1, \cdots, x_n$ be an ordered sample of size n from a population with continuous cumulative distribution function $F(x)$, and define $\omega_n = \frac{1}{2}\sum_{i=1}^{n+1}|F(x_i) - F(x_{i-1}) - 1/(n+1)|$, where $x_0 = -\infty$, $x_{n+1} = +\infty$. An explicit formula for the

moments of ω_n is obtained; the cumulative distribution function of ω_n is

$$G(x) = 1 + \sum_{q=0}^{n-r-1}\sum_{p=0}^{q}(-1)^{q-p+1}\binom{n}{p}\binom{n+1}{q+1}\binom{n+q-p}{n}$$
$$\times \{(n-q)/(n+1)\}^p\{(n-q)/(n+1) - x\}^{n-p},$$

where r is the nonnegative integer determined by

$$r/(n+1) \leq x < (r+1)/(n+1).$$

Also, $E(\omega_n) \sim 1/e$, $\sigma^2(\omega_n) \sim (2e-5)/e^2 n$, and $[\omega_n - E(\omega_n)]/\sigma(\omega_n)$ is asymptotically normal. If x has cumulative distribution function $G(x) \neq F(x)$, then $[s_n - E(\omega_n)]/\sigma(\omega_n) \to +\infty$ in probability, where $s_n = \frac{1}{2}\sum_{i=1}^{n+1}|F(x_i) - F(x_{i-1}) - 1/(n+1)|$, so that the test of goodness of fit based on ω_n is consistent.

D. Blackwell (Stanford University, Calif.).

Stein, Charles. Unbiased estimates with minimum variance. Ann. Math. Statistics **21**, 406–415 (1950).

Let $p(x|\theta)$, $\theta \epsilon \Omega$, be a class of density functions with respect to a measure μ defined on a Borel field of subsets of X, and let $g(\theta)$ be a numerical function of θ. If $p(x|\theta_0) > 0$ for all x, $\int (p^2(x|\theta)/p(x|\theta_0))d\mu < \infty$ for all θ, and there is an unbiased estimate for g with finite variance at θ_0, then there is exactly one, f^*, of minimum variance at θ_0. Under special circumstances, f^* has the form $\int p(x|\theta)d\lambda(\theta)$ for appropriately chosen $\lambda(\theta)$. As examples, f^* is described when (1) x is rectangular over $(\theta, \theta+1)$, $-N < \theta < N-1$, $\theta \neq \theta_0$, and x is rectangular over $(-N, N)$ under θ_0, (2) x consists of n independent variables rectangular over $(0, \theta)$, and (3) x consists of n independent normal variables with mean θ and unit variance. *D. Blackwell* (Stanford University, Calif.).

Lehmann, E. L., and Stein, Charles. Completeness in the sequential case. Ann. Math. Statistics **21**, 376–385 (1950).

Necessary conditions are found for the uniqueness of unbiased estimates obtained from sequential sampling schemes based on a class of distribution families to which attention was called by Blackwell [same Ann. **18**, 105–110 (1947); these Rev. **8**, 478]. By their aid it is shown that for the normal distribution of undetermined mean and unit variance the sample size must be fixed for uniqueness. Necessary and sufficient conditions are also given for the uniform family with lower bound at zero, the Poisson family, and the binomial family. The last of these has already been the subject of considerable study [see Wolfowitz, same Ann. **17**, 489–493 (1946); **18**, 131–135 (1947); these Rev. **8**, 477; Savage, ibid. **18**, 295–297; these Rev. **9**, 152], but sufficiency of the condition given in this case is new.

L. J. Savage (Chicago, Ill.).

Hemelrijk, J., and van der Vaart, H. R. The use of unilateral and bilateral critical regions in the testing of hypotheses. Statistica, Rijswijk **4**, 54–66 (1950). (Dutch. English summary)

[Second author's name misprinted van der Vaar in title.] This paper offers an exposition of the principles of the Neyman-Pearson theory, particularly with respect to the use of critical regions indicated in the title. The arguments are illustrated by a few simple examples (without showing numerical data), including a discussion of the distribution of Cimex columbarius and lectularius. *A. A. Bennett.*

Gardner, Robert S. A non-parametric test of the hypothesis that two bivariate samples come from the same population. Naval Ordnance Test Station, Inyokern, Calif. Appendix from Tech. Memo. 4542-33, i+4 pp. (Undated).

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be n independent random vectors with a common (unknown) continuous cumulative distribution function $F(x, y)$ which are independent of m random vectors $(\xi_1, \eta_1), \cdots, (\xi_m, \eta_m)$. A test is proposed of the hypothesis that the (ξ_j, η_j) are independent with common cumulative distribution function $F(x, y)$. It is based on results of Wilks, Wald, and Tukey concerning the distribution of coverages [see Wilks, Bull. Amer. Math. Soc. **54**, 6–50 (1948); these Rev. **9**, 601]. The test involves the number z of points (ξ_j, η_j) contained in a random set S which depends on the (X_i, Y_i) and $k \leq n$ arbitrary functions. The distribution of z when the hypothesis is true and its asymptotic form for $m \to \infty$ are obtained. The power of the test is not considered. [There are misprints in the expressions for $p(u)$ and $E(z)$.] *W. Hoeffding.*

Tocher, K. D. Extension of the Neyman-Pearson theory of tests to discontinuous variates. Biometrika **37**, 130–144 (1950).

The difficulties which arise in connection with the size of a test involving discontinuous variates may be avoided by the use of a randomized test. Necessary modifications in the Neyman-Pearson fundamental lemma and the treatment of similar regions are discussed. The methods are applied to tests of hypotheses concerning 2×2 tables and the problem of comparing two Poisson variables.
G. E. Noether (New York, N. Y.).

Hald, A. Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point. Skand. Aktuarietidskr. **32**, 119–134 (1949).

New tables are given to facilitate the solution of the indicated problem. These should be more useful than those previously given by Pearson and Lee [Pearson, Tables for Statisticians and Biometricians, Part I, 3d ed., Biometrics Laboratory, University College, London, 1930, Table XI; Part II, 1st ed., 1931, Table XII] or by R. A. Fisher [British Association for the Advancement of Science, Mathematical Tables, vol. I, 1st ed., pp. xxv–xxxv, 1931] [cf. also Cohen, J. Amer. Statist. Assoc. **44**, 518–525 (1949); these Rev. **11**, 258]. Further tables are presented to solve in a similar manner the maximum likelihood equations from a "censored" normal distribution (one for which the frequency of observations below the truncation point is known).
D. G. Chapman (Seattle, Wash.).

Shenton, L. R. Maximum likelihood and the efficiency of the method of moments. Biometrika **37**, 111–116 (1950).

An expansion is given, based on a formula of Fisher, for the efficiency of the method of estimating the s parameters of a distribution from the first s sample moments. As an example, the author discusses the efficiency of the method for estimating the parameters α, β for a variable x which for fixed p has a binomial distribution, where p has density $Kp^{\alpha-1}(1-p)^{\beta-1}$. *D. Blackwell.*

Gulliksen, Harold, and Wilks, S. S. Regression tests for several samples. Psychometrika **15**, 91–114 (1950).

Let $(Y_{ki}, X_{1ki}, \cdots, X_{Hki})$, $i = 1, \cdots, n_k$, be a random sample of size n_k, $k = 1, \cdots, K$, from a multivariate population such

that Y is normally distributed with variance σ_k^2 around the regression plane, $\alpha_k + \sum_{h=1}^{H} \beta_{hk} X_h$, of Y on $X_1, \cdots, X_H$, where values of the X_h are regarded as "fixed." The K samples are assumed to be mutually independent. Neyman-Pearson likelihood ratio criteria are given for the following hypotheses: H_A*: $\sigma_1^2 = \cdots = \sigma_K^2$; H_B*: $\beta_{h1} = \cdots = \beta_{hK}$, $h = 1, \cdots, H$, given H_A*; H_C*: $\alpha_1 = \cdots = \alpha_K$, given H_B*; H_S*: H_A*, H_B*, and H_C* are true simultaneously. The sampling theory of each criterion, when the corresponding hypothesis is true, is given for large samples. An illustrative problem is included. *D. F. Votaw, Jr.*

✹**Wald, Abraham.** Statistical Decision Functions. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. ix+179 pp. $5.00.

In a series of papers [of which Econometrica **15**, 279–313 (1947); these Rev. **9**, 454 is the most complete] Wald has developed the theory of statistical decision functions. This book is primarily a unification and extension of the author's earlier work. The general model is that of a statistical decision problem, consisting of a space Ω of probability distributions ω of a (possibly infinite-dimensional) chance variable x, a space D of decisions d available to the statistician, and a numerical, usually nonnegative, function $W(\omega, d)$, the loss to the statistician when ω is the distribution of x and he makes decision d. The aim of the statistician is to choose d so that W is in some sense minimized. Practically all statistical problems, including estimation, testing hypotheses, and the design of experiments, can be formulated in this way. The structure of various rather general statistical problems is analysed in detail; a partial summary of contents by chapters is given below.

1. The general statistical decision problem: definitions and preliminary discussion. The general problem is formulated, essentially as above, and the concepts of a priori distribution ξ over Ω, randomized decision procedure η (a probability distribution over D) and risk function $R(\xi, \eta) = \int\int W(\omega, d)d\xi d\eta$ are introduced. An η_0 is a Bayes solution if there is a ξ with $R(\xi, \eta_0) = m(\xi) = \inf_\eta R(\xi, \eta)$, a minimax solution if $\sup_\xi R(\xi, \eta) = M(\eta)$ is minimized for $\eta = \eta_0$ and is admissible if $M(\xi, \eta) \leq M(\xi, \eta_0)$ for all ξ implies equality for all ξ. It is noted that the structure of the decision problem is formally identical with that of the two-person zero-sum game considered by von Neumann and Morgenstern [Theory of Games and Economic Behavior, 2d ed., Princeton University Press, 1947; these Rev. **9**, 50] (except that the latter authors consider only the case where Ω and D are finite, a restriction removed by other writers).

2. Zero-sum two-person games with infinitely many strategies. A statistical decision problem or game is strictly determined if $\sup_\xi m(\xi) = \inf_\eta M(\eta)$. If the space Ω is conditionally compact with respect to the metric

$$\rho(\omega_1, \omega_2) = \sup_d |W(\omega_1, d) - W(\omega_2, d)|,$$

identifying ω_1 and ω_2 if $\rho(\omega_1, \omega_2) = 0$, the game is shown to be strictly determined. This result is an extension of the corresponding theorem of von Neumann and Morgenstern for the case in which Ω and D are finite, and uses their result in its proof. A class C of strategies η is complete if for every η_0 there is an $\eta_1 \epsilon C$ with $R(\xi, \eta_1) \leq R(\xi, \eta_0)$ for all ξ. The result about strictly determined games yields the fact that under certain conditions the class of Bayes solutions is complete and that, under weakened conditions, the somewhat wider class of all strategies η_0 such that $\sup_\xi [m(\xi) - R(\xi, \eta_0)] = 0$, called minimal strategies, is complete.

3. Development of a general theory of statistical decision functions. The detailed structure of Ω, D, W is discussed and various assumptions are formulated. The assumptions are too detailed to be given here; the concepts involved include a sequence of numerical chance variables $x = x_1, x_2, \cdots$, which represent the possible experiments available to the statistician, and whose joint distribution is determined by ω, the possible sampling plans, i.e., designs of the experiment, available, the cost of each design as a function of x, the class of terminal actions a available after sampling is concluded, and the loss from terminal action a as a function of ω. The assumptions are shown to guarantee that the results of chapter 2 can be applied, so that, for instance, the statistical decision problem is strictly determined, the statistician has a minimax strategy, and the class of minimal strategies is complete.

4. Properties of Bayes solutions when the chance variables are independently and identically distributed and the cost of experimentation is proportional to the number of observations. The content of this chapter is indicated by its title, and as indicated by the author, is essentially that of an article by him and Wolfowitz [Ann. Math. Statistics 21, 82–99 (1950); these Rev. 11, 529].

5. Application of the general theory to various special cases. Various fixed experiment (i.e., no choice of experiments is available) and sequential (i.e., the order of experiments is prescribed) problems are considered. For fixed experiment problems in which each of Ω, D contains two elements, the class of Bayes solutions is the class of likelihood ratio tests. The fixed experiment minimax estimates for the mean of (a) a binomial variable and (b) a variable with range $(0, 1)$ are described, with the loss taken as the squared error of estimate. Sequential examples discussed include a two-sample procedure for deciding whether the mean of a normal variable with unit variance is Δ or $-\Delta$, and the sequential point estimation of the mean of a rectangular variable with unit range.   *D. Blackwell.*

Chakrabarti, M. C.   A note on balanced incomplete block designs.   Bull. Calcutta Math. Soc. 42, 14–16 (1950).

Three results are proved for a balanced incomplete block design with parameters v, k, b, r, λ: (i) $v \le b$; (ii) if v is even, no design can exist unless $(r - \lambda)$ is a perfect square; (iii) if l_{ij} is the number of varieties common to the ith and jth block,

$$|l_{ij}| = \begin{cases} 0, & b > v, \\ r^2(r-\lambda)^{v-1}, & b = v. \end{cases}$$

The first two results have been proved previously.
                          W. G. Cochran (Raleigh, N. C.).

Kelley, J. L.   Convergence in topology.   Duke Math. J. 17, 277–283 (1950).

This paper contains a systematic extension and simplification of results of G. Birkhoff [Ann. of Math. (2) 38, 39–56 (1937)] and Tukey [Convergence and Uniformity in Topology, Princeton University Press, 1940; these Rev. 2, 67] on Moore-Smith convergence in topology. A function S defined on a directed set D is called a "net" and is denoted by S, $>$ where "$>$" is the partial ordering in D. The value of S on $n \epsilon D$ is denoted by S_n. A net S, $>$ on D is said to be "eventually in X" if there exists $p \epsilon D$ such that $S_n \epsilon X$ for

Sukhatme, P. V.   Efficiency of sub-sampling designs in yield surveys.   J. Indian Soc. Agric. Statistics 2, 212–228 (1950).

Storch, J. M.   The use of moving averages for the analysis of an economic graph.   Statistica, Rijswijk 4, 37–53 (1950).   (Dutch.   English summary)

The author considers certain linear operators (which he calls "linors") upon successive entries in a numerical table, such as $\delta^2 = (1, -2, 1)$ and $S^2 = (1, 2, 1)$. The effect of these and combinations of these upon trend and cyclic variation is considered. The author concludes that suitably chosen linors systematically applied may serve to split up a given time series into trend, cyclic variation, and random residue. Smoothness and fitting of such moving averages are studied.
                       A. A. Bennett (Providence, R. I.).

Plackett, R. L.   Some theorems in least squares.   Biometrika 37, 149–157 (1950).

Let $A = \{a_{ij}\}$, $i = 1, \cdots, n$, $j = 1, \cdots, s$, be an $(n \times s)$ matrix of known quantities, let $\theta = \{\theta_1, \cdots, \theta_s\}$ be an s-dimensional vector of unknown parameters θ_i, $i = 1, \cdots, s$, and let $x = \{x_1, \cdots, x_n\}$ be an n-dimensional vector of observations. Further let (1) $Ex = A\theta$ and (2) $E(x - A\theta)(x - A\theta)' = \sigma^2 I_n$ hold, where I_n is the $n \times n$ unit matrix. Then it is known (according to Gauss and others) that (3) $\theta^* = C^{-1}A'x$ is a linear unbiased estimate of θ having minimum variance, provided that A has rank s. In (3), $C = A'A$.

The author first raises the following question. How must the method of least squares be modified if the rank of the matrix A is less than the number of parameters? In this case (3) becomes meaningless and there does not exist a linear unbiased estimate of θ having minimum variance. The author shows that if in addition to (1) there exists a set of constraints on θ, expressed by (4) $y = B\theta$, where y is a nonrandom known vector, then under certain conditions on the matrices A and B, the linear unbiased estimate having minimum variance is $\theta^* = (A'A + B'B)^{-1}(A'x + B'y)$. The conditions on A and B are too involved to be given in this review. The second problem treated by the author deals with the question of how one can revise the estimates of parameters, the associated covariance matrix, and the sum of the squares of residuals, if one or more additional observations become available. The author gives a number of useful formulae in this connection and gives a numerical illustration which is worked out in some detail.
                         B. Epstein (Detroit, Mich.).

Tárczy-Hornoch, A.   Über die Zurückführung der Methode der kleinsten Quadrate auf das Prinzip des arithmetischen Mittels.   Österreich. Z. Vermessgswes. 38, 13–18 (1950).

TOPOLOGY

$n > p$. A net T, $>_1$ on E is called a "subnet" of the net S, $>$ on D if there is a function N on E to D such that (a) if $p \epsilon D$ then N, $>_1$ is eventually in the set of all $q \epsilon D$ for which $q > p$ and (b) $T_i = S_{Ni}$ for all $i \epsilon E$. Subnet thus generalizes the notion of subsequence and provides the main point of the present treatment. A net S, $>$ in a topological space is said to converge to a point s if S, $>$ is eventually in every neighborhood of s. A typical theorem proved here is that a topological space X is compact if and only if every net in X has a subnet which converges to a point of the space.
                      C. E. Rickart (New Haven, Conn.).

Sierpiński, Wacław. Sur les relations entre quelques propriétés fondamentales des espaces topologiques. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **40** (1947), 66–78 (1948). (French. Polish summary)

Let S be a T_1 space. The author studies relations among the following properties concerning S. (H): S is a Hausdorff space; (A): S satisfies the first axiom of countability; (B): S possesses a denumerable base; (C): no infinite sequence can sequentially converge to two distinct limits; (D): S is a denumerable set. It is shown that $A \cdot D \to B$ and $A \cdot C \to H$. He then shows the existence or nonexistence (by examples or easy deductions from the relations given in the preceding sentence and the facts that $B \to A$ and $H \to C$) of T_1 spaces having the above properties (or negations) conjointly. All the thirty-two possibilities are studied. If G is the property that $E \subset S$ and $a \varepsilon E - \{a\}$ implies a is the sequential limit of a sequence of elements different from a, then clearly $A \to G$. An example of a denumerable Hausdorff space which does not enjoy G is given. *H. Tong* (Paris).

Katětov, Miroslav. On nearly discrete spaces. Časopis Pěst. Mat. Fys. **75**, 69–78 (1950). (English. Czech summary)

This paper generalizes, extends, and illuminates results previously obtained by the author [Rec. Math. [Mat. Sbornik] N.S. **21**(63), 3–12 (1947); these Rev. **9**, 98] and by the reviewer [Duke Math. J. **10**, 309–333 (1943); these Rev. **5**, 46]. A one-to-one continuous mapping φ of a Hausdorff space X into a Hausdorff space Y is said to be an L-mapping if $\varphi(x)$ is isolated in Y whenever x is isolated in X. A Hausdorff space P is called nearly discrete if every L-mapping of a Hausdorff space onto P is a homeomorphism. A semi-regular [regular, completely regular] space P is called nearly (SR) [nearly (R), nearly (CR)] discrete if every L-mapping of a semi-regular [regular, completely regular] space onto P is a homeomorphism. A number of diverse results concerning such spaces are proved, of which the following are typical. The following properties of a semi-regular space P are equivalent: (a) P is nearly (SR) discrete; (b) P is nearly (R) discrete; (c) P is nearly (CR) discrete; (d) if Q and Q' are complementary subsets of P such that points isolated in Q or Q' are already isolated in P, then Q and Q' are open; (e) if Q_1 and Q_2 are subsets of P such that points isolated in Q_1 or in Q_2 are already isolated in P, then $Q_1 \cap Q_2$ enjoys this property also. Every Hausdorff [regular] space is an L-image of a nearly discrete [nearly (R) discrete] space. Let S be a completely regular space, and let P be a dense subspace of S which is nearly (R) discrete. Then S is nearly (R) discrete if and only if $S \subset \beta P$ and $S \cap P'$ contains no subset dense-in-itself. Every countably compact subspace of a nearly (R) discrete space is finite. Any nearly (R) discrete space P may be imbedded in βI, where I is a discrete space such that $|I| = |D|$, D being any dense subset of P. A number of examples, illustrating the various possibilities involved, are also given. *E. Hewitt* (Seattle, Wash.).

De Giorgi, Ennio. Costruzione di un elemento di compattezza per una successione di un certo spazio metrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 302–304 (1950).

There is shown, perhaps more explicitly than usual ("effectively"), how to obtain a limit point, in the space of closed subsets (under Hausdorff's metric) of a compactum, of any given sequence of closed subsets. *R. Arens*.

Yajima, Takeshi. On a local property of absolute neighbourhood retracts. Osaka Math. J. **2**, 59–62 (1950).

According to a well-known theorem of Kuratowski [Fund. Math. **24**, 269–287 (1935)], a compact metric space Y is an ANR (absolute neighborhood retract) if, whenever f is a mapping into Y of a closed subset A of a metric space X, then there is an extension f^* of f to an open neighborhood E of A in X. In this note it is shown that this characterization can be localized in the following fashion. A space Y is said to have property (L) if for each point $p \varepsilon Y$ and each neighborhood U of p there is a neighborhood $V \subset U$ of p such that any mapping of a closed subset A of a metric space X into V can be extended over X with respect to U. It is proved that a compact metric space is an ANR if and only if it has property (L). A number of consequences are derived, including the theorem that a compact metric space, each point of which has a neighborhood homeomorphic to an open subset of a compact ANR, is itself an ANR. *E. G. Begle* (New Haven, Conn.).

Borsuk, Karol. On the imbedding of n-dimensional sets in $2n$-dimensional absolute retracts. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 112–116 (1950).

Let $m(n)$ denote the smallest integer such that any separable metric space of dimension n can be imbedded in an AR (absolute retract) of dimension $m(n)$. The Menger-Nöbeling theorem shows that $m(n) \leq 2n + 1$. Here it is shown that $m(n) \leq 2n$ provided that $n > 0$. In fact, a slightly stronger theorem is proved, in that the $2n$-dimensional AR can be required to be a subset of the $(2n+1)$-dimensional Euclidean space. *E. G. Begle* (New Haven, Conn.).

Kuratowski, Casimir. Quelques généralisations des théorèmes sur les coupures du plan. Fund. Math. **36**, 277–282 (1949).

Let S^2 be the 2-sphere. If $E \subset S^2$, and p, $q \varepsilon S^2 - E$, then E cuts S^2 between p and q if $S^2 - E$ contains no continuum joining p and q. Let $A_0, \cdots, A_{n-1}$ ($n \geq 3$) be a sequence of sets, and define (1) $X = A_i + \cdots + A_{n-1}$, (2) $B_k = A_{k+1} + \cdots + A_{k+n-1}$, (3) $C_k = A_{k+1} + \cdots + A_{k+n-2}$, (4) $P = C_0 \cdots C_{n-1}$, reducing the subscripts modulo n in (2) and (3). Suppose that the sets $C_0, \cdots, C_{n-1}$ are connected. Theorem I: If (i) p, $q \varepsilon S^2 - X$, (ii) none of the sets B_k cut S^2 between p and q, and (iii) $P \neq 0$, then X does not cut S^2 between p and q. Theorem II: If (i) p, q, and r are different points of $S^2 - X$, and (ii) none of the sets B_k cut S^2 between any two of the points p, q, r, then it is false that X cuts S^2 between every two of the points p, q, r. For $n = 3$, these are theorems of Eilenberg [Fund. Math. **26**, 61–112 (1936), pp. 78–79]. *E. E. Moise* (Princeton, N. J.).

✶**König, Dénes.** Theorie der endlichen und unendlichen Graphen. Kombinatorische Topologie der Streckenkomplexe. Chelsea Publishing Co., New York, N. Y., 1950. ix+258 pp.

Photographic reproduction of a book originally published in 1936 by the Akademische Verlagsgesellschaft, Leipzig.

Medgyessy, Paul. Sur la structure des réseaux finis, cubiques et coloriés. Mathesis 59, 173–176 (1950).

The author discusses planar cubic graphs in which the edges are coloured in three colours so that no two of the same colour have a common vertex. Such graphs are called

"coloured." He describes a process whereby any coloured planar cubic graph can be derived from a "fundamental" one. The fundamental graph consists of two vertices joined by three edges of different colours.   *W. T. Tutte.*

✱**Reidemeister, Kurt. Einführung in die kombinatorische Topologie.** Chelsea Publishing Co., New York, N. Y., 1950. x+209 pp. $6.00.

Photographic reproduction of a book originally published in 1932 by Vieweg, Braunschweig.

Shapiro, Arnold. Cohomologie dans les espaces fibrés. C. R. Acad. Sci. Paris **231**, 206–207 (1950).

Let D and E be locally compact and paracompact Hausdorff spaces. Assume that E is a fibre bundle over D relative to a projection $p: E \to D$ with compact fibres F. Let $\mathfrak{D}$ be a fundamental grating on D and $\mathfrak{S}$ be a fine grating on E. If one defines new supports for $\mathfrak{D}$ by $s'(a) = p^{-1}(s(a))$ for each $a \varepsilon \mathfrak{D}$, he obtains a grating $\mathfrak{D}'$ on E. Denote by $\mathfrak{D}*\mathfrak{S}$ the separated and complete grating on E associated with the grating $\mathfrak{D}' \otimes \mathfrak{S}$. The ring $\mathfrak{D}*\mathfrak{S}$ contains a subring isomorphic to $\mathfrak{D}_A = \mathfrak{D} \otimes A$ which will be identified with $\mathfrak{D}_A$. The author announces the following general theorem. If $H(F, A)$ is a free module over A, and if $\pi_1(D)$ operates trivially on $H(F, A)$, then there exists a $\mathfrak{D}_A$-isomorphism β of $L = \mathfrak{D} \otimes H(F, A)$ into $\mathfrak{D}*\mathfrak{S}$ (both considered as $\mathfrak{D}_A$-modules) which has the following properties: (a) $\beta(L)$ is stable under the coboundary operator d; (b) $H(\beta(L))$ is isomorphic to $H(\mathfrak{D}*\mathfrak{S}) = H(E, A)$; and (c) $d\beta(\mathfrak{D} \otimes H^n(F, A))$ is contained in $\sum_{m \leq n-1} \beta(\mathfrak{D} \otimes H^m(F, A))$. As an application of this theorem, the author sketches a proof of a conjecture of Montgomery and Samelson which is as follows. If E is the n-dimensional Euclidean space and if $p: E \to D$ is a fibre mapping with compact fibres F, then p is a homeomorphism. For lower dimensions, namely, $n \leq 4$, this has been proved by G. S. Young [Proc. Amer. Math. Soc. **1**, 215–223 (1950); these Rev. **11**, 610].   *S. T. Hu* (Princeton, N. J.).

Keesee, John W. Finitely-valued cohomology groups. Proc. Amer. Math. Soc. **1**, 418–422 (1950).

Il s'agit de la cohomologie d'Alexander-Kolmogoroff d'un espace topologique X; elle coïncide avec la cohomologie de Čech si X est compact. L'auteur se sert des cochaînes de Spanier pour obtenir cette cohomologie [Ann. of Math. (2) **49**, 407–427 (1948); ces Rev. **9**, 523]. Il prouve que les cochaînes qui ne prennent qu'un nombre fini de valeurs conduisent aussi à la cohomologie de Čech quand X est compact; c'est d'ailleurs évident quand on pense à la définition de Čech par les recouvrements ouverts finis. Le théorème est formulé et démontré pour la cohomologie relative.   *H. Cartan* (Paris).

Reeb, Georges. Variétés de Riemann dont toutes les géodésiques sont fermées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 324–329 (1950).

A Riemannian manifold is said to be of type Γ if all its geodesics are closed, and their length depends continuously on the initial point and direction. It follows that the space V_{2n-1} of unit vectors is fibered, with fiber a circle, and with base space a manifold W_{2n-2}, and that Cartan's invariant integral determines a differential 1-form π whose derivative $d\pi$ can be considered as a form Ω on W_{2n-2}, of maximum rank (i.e., $(d\pi)^{n-1} \neq 0$); Ω is the characteristic cocycle. Considering the known relations between Betti numbers of bundle and base for a fibration into circles, the author shows: if n is even, if the Betti numbers b_p of an orientable manifold V_{2n-1} vanish for even dimensions between 0 and $n-1$, and if the sum of the Betti numbers between 0 and $n-1$ is odd, then V_{2n-1} does not admit a fibration into circles whose characteristic class Ω satisfies $\Omega^{n-1} \neq 0$. From this one derives: a manifold V_n with Poincaré-polynomial $1 + t^q + t^{n-q} + t^n$ (n even, q odd >1) is not of type Γ. The product of two odd spheres falls in this category. Finally, the first Betti number of a manifold of class Γ is shown to be 0 as a consequence of the fact that the fibers of V_{2n-1} are rationally ~ 0.   *H. Samelson* (Ann Arbor, Mich.).

GEOMETRY

Cavallaro, Vincenzo G. Relazioni areali di triangoli podari e omopodari. Euclides, Madrid **10**, 65–68 (1950).

The author discusses some theorems in plane Euclidean geometry. If triangle ABC is a fixed triangle and M is an arbitrary point, the triangle $\triangle_M$ whose vertices are the orthogonal projections of M upon the sides of triangle ABC, and the triangle $\triangle'_M$, whose sides pass through the vertices of triangle ABC and are parallel to those of $\triangle_M$, are such that the area of triangle ABC is the geometric mean between the areas of $\triangle_M$ and $\triangle'_M$, according to a theorem of Pilatte. If the power of a point M is defined as the quotient of the areas of $\triangle_M$ and triangle ABC, multiplied by four times the square of the circumradius of triangle ABC, the author obtains the power of the point for the various famous points of the triangle ABC. Similar expressions are obtained for the areas of $\triangle_M$ and $\triangle'_M$ where M represents any one of these points. If S_O, S_I, S_J represent the areas of the triangles $\triangle_M$ where M is at the circumcenter, incenter, and Nagel point, then $S_J/S_I + S_I/S_O = 2$. Many other relations between these expressions are obtained.   *J. De Cicco* (Chicago, Ill.).

Schuh, Fred. The volume of a tetrahedron and the radius of the circumscribed sphere expressed in terms of the edges. Nieuw Tijdschr. Wiskunde **38**, 2–6 (1950). (Dutch)

Thébault, V. Sur les points qui divisent dans un même rapport quatre segments arbitraires de l'espace. Mathesis **59**, 86–89 (1950).

The author provides a very elementary proof of a theorem due to R. Deaux [Mathesis **46**, 147–149 (1932), p. 149] to the effect that in general three planes exist cutting 4 arbitrary segments of three-space in the same ratio. Conditions are found under which at most two, one, or no such planes exist. Consideration is given to three cases in which the two quadruples of endpoints of the 4 segments are in special relations to each other.   *L. M. Blumenthal.*

✱**Bravais, A. On the Systems Formed by Points Regularly Distributed on a Plane or in Space.** The Crystallographic Society of America, Cambridge, Mass., 1949. viii+113 pp. $3.90. (Copies may be ordered from the Secretary-Treasurer, William Parrish, Crystallographic Laboratory, Philips Laboratories, Inc., Irvington-on-Hudson, N. Y.)

[Translated from J. École Polytech. (1) **19**, cahier 33, 1–128 (1850).] A lattice in Euclidean 3-space is the set of all transforms of a point under the group generated by three independent translations. This infinite group may be the whole symmetry group of the lattice, or there may be also some reflections and rotations. When lattices are classified

according to the nature of the whole symmetry group, there are found to be just fourteen distinct types. It is highly appropriate that A. J. Shaler has translated the classical memoir in which Bravais expounded this discovery. There are few expositions which retain such freshness and validity a hundred years after their first appearance. The main development is expressed in 99 theorems, followed by a remarkable section "On polar lattices." The polar of a given lattice, like the "reciprocal" [Niggli, Handbuch der Experimentalphysik, vol. 7.1, Leipzig, 1928, p. 11] is similar to the lattice whose three generating vectors are the vector products of pairs among the generating vectors of the original lattice. Bravais mentions the Seeber-Gauss representation of a lattice by a ternary quadratic form [Gauss, Werke, v. 2, p. 192] and shows that the polar lattice is represented by a suitable multiple of the adjoint form.

The translation into English has been very well done, except that on pages 98–100 the words "threefold, twofold, adjoining" should have been "ternary, binary, adjoint," that the author is A. Bravais (not "M. A. Bravais"), and that "parallelepiped" should not be spelt with an o.

H. S. M. *Coxeter* (Toronto, Ont.).

✶Hlavatý, V. **Úvod do neeuklidovské geometrie.** [Introduction to Non-Euclidean Geometry]. 2d ed. Jednota Československých Matematiků a Fysiků, Prague, 1949. 227 pp.

This book for the layman contains only nonessential changes and complements to the first edition [1926]. The book does not contain the axiomatics of geometry. It starts from the elementary notions of the analytic geometry of the metric plane and deals with non-Euclidean geometry as the study of properties invariant with respect to a certain group of linear transformations. From this point of view it deals with the one-dimensional non-Euclidean variety and the varieties of the hyperbolic, elliptic, and parabolic plane.

F. *Vyčichlo* (Prague).

Petrov, G. **La méthode projective de Monge dans l'espace elliptique.** Časopis Pěst. Mat. Fys. 75, 27–42 (1950). (French. Czech summary)

En continuant ses recherches sur la méthode projective [dans une publication bulgare pas disponible] l'auteur, dans ce travail, expose en détail la construction de cette méthode dans l'espace elliptique en utilisant de nouveau le modèle de Cayley-Klein. Dans l'espace il choisit une sphère réelle au rayon unité comme substitut réel de la surface absolue et il prend deux plans perpendiculaires diamétraux comme plans projectifs. On obtient les projections d'un point P en projettant P orthogonalement sur les plans de projection. La projection est en même temps elliptique et euclidienne orthogonale.   H. A. *Lauwerier* (Amsterdam).

Convex Domains, Extremal Problems

Sperner, Emanuel. **Konvexität bei Ordnungsfunktionen.** Abh. Math. Sem. Univ. Hamburg 16, nos. 3–4, 140–154 (1949).

Given an affine n-space R_n over a field or skew field K ($n \geq 2$). Let h, k denote [$(n-1)$-dimensional] hyperplanes, and let α, β, γ denote points of R_n. If parallel coordinates $x_1, \cdots, x_n$ are introduced, each h is given by an equation $\mathfrak{u}\mathfrak{x} = u_0 + \sum \mathfrak{u}_i x_i = 0$; here $\mathfrak{u} = (u_0, u_1, \cdots, u_n)$ is a coordinate

vector of h, while $\mathfrak{x} = (1, x_1, \cdots, x_n)$ is the coordinate vector of a point moving in h. Given two hyperplanes h, k and two points α, β with the coordinate vectors $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{x}$, $\mathfrak{y}$, respectively. If $\mathfrak{u}\mathfrak{x} \cdot \mathfrak{v}\mathfrak{y} \cdot \mathfrak{v}\mathfrak{y} \neq 0$, their cross-ratio is defined by $D(h, k \,|\, \alpha, \beta) = (\mathfrak{u}\mathfrak{y})^{-1} \cdot (\mathfrak{u}\mathfrak{x}) \cdot (\mathfrak{v}\mathfrak{x})^{-1} \cdot (\mathfrak{v}\mathfrak{y})$. An order function $h(\alpha)$ is a mapping of the pairs h, α on the numbers 0, ± 1 which satisfies two conditions: (1) $h(\alpha) = 0$ if and only if $\alpha \subset h$; (2) if α, β, γ are collinear and h, k contain γ but neither α nor β, then $h(\alpha)h(\beta)k(\alpha)k(\beta) = 1$. The hyperplane h is said to lie between α and β if $h(\alpha)h(\beta) = -1$. A point γ collinear with α and β lies between these two points if some h through γ lies between them. A point set is convex if it contains with any two points all the points between them. Finally, the order function $h(\alpha)$ is called convex if, for each h, the set of all the α with $h(\alpha) = 1$ and that of all α with $h(\alpha) = -1$ are convex.

A semi-ordering of K divides the nonzero elements $a, b, \cdots$ of K into two classes, a positive and a negative one, such that $\operatorname{sgn} ab = \operatorname{sgn} a \cdot \operatorname{sgn} b$. In the trivial semi-ordering, $a \neq 0$ implies $a > 0$. Every order function in R_n induces a semi-ordering of K by means of rule A:

$$\operatorname{sgn} D(h, k \,|\, \alpha, \beta) = h(\alpha)h(\beta)k(\alpha)k(\beta).$$

Conversely, every semi-ordering determines an order function through rule B: Choose a definite coordinate vector $\mathfrak{u}$ for each h and define $h(\alpha) = \operatorname{sgn} (\mathfrak{u}\mathfrak{x})$, where $\mathfrak{x}$ denotes the coordinate vector of α. An order function obtained by means of this rule is called normal. Application to it of rule A leads back to the original semi-ordering. The following results may be mentioned. A normal order function is convex if and only if either K is ordered by the induced (nontrivial) semi-ordering or if K is the prime field of three elements with the semi-ordering $1 > 0$, $-1 < 0$. Suppose $h(\alpha)$ induces a nontrivial semi-ordering of K. Then $h(\alpha)$ is normal if and only if $h(\alpha)h(\beta) = 1$ for any two points $\alpha \neq \beta$ and any hyperplane h which does not meet the straight line through them. Finally, criteria are derived for an order function to induce a semi-ordering which can also be induced by a normal convex order function.   P. *Scherk*.

Fejes Tóth, László. **Über den Brunn-Minkowskischen Satz.** Mat. Lapok 1, 211–217 (1950). (Hungarian. Russian and German summaries)

The note deals (a) with a theorem of Brunn, (b) with a theorem of Brunn-Minkowski. (a) Schwarz symmetrization preserves convexity. (b) Let T_0 and T_1 be two convex domains in a plane and T_λ the linear set of mean convex domains defined in the usual way. Denote by $T(\lambda)$ the area of T_λ; then $(T(\lambda))^{\frac{1}{2}}$ is a convex function of λ. The author's proof of these theorems is slightly different from a proof given by Blaschke [Kreis und Kugel, Veit, Leipzig, 1916, § 21]. In particular, instead of using the decrease of the surface area under Schwarz symmetrization, he uses the same property of the moment of inertia; this fact is much easier to prove.   G. *Szegö* (Stanford University, Calif.).

Hlawka, Edmund. **Über Integrale auf konvexen Körpern. I.** Monatsh. Math. 54, 1–36 (1950).

Let B be a convex body in R_n and let the Gaussian curvature of its surface have a positive lower bound. Suppose the origin is an inner point of B and that $f(x)$ is the distance-function of B, where $x = (x_1, \cdots, x_n)$. Write dx for the element of volume and $lx = l_1 x_1 + \cdots + l_n x_n$, where $l = (l_1, \cdots, l_n)$. The author discusses integrals of the type $\int_B \Phi(f(x)) e^{ilx} dx$ subject to differentiability conditions on the surface of B

and on the arbitrary function $\Phi(u)$, using the method of stationary phase.

For given unimodular matrix A, vector y, and real number $u \geq 0$, denote now by $\Phi(y, u)$ the number of integer vectors x satisfying $f^2(A(x-y)) \leq u$, i.e., $\Phi(y, u)$ is the number of lattice points in a body, u, times B displaced to Ay. Then, if $\mathfrak{I}$ is the volume of B, we have as $u \to \infty$,

(i) $\qquad\qquad \Phi(y, u) = \mathfrak{I}u^{m/2} + O(u^{m(m-1)/2(m+1)})$,

(ii) $\qquad\qquad \Phi(0, u) = \mathfrak{I}u^{m/2} + \Omega(u^{(m-1)/4})$.

Let $\lambda(u) \nearrow \infty$ as $u \nearrow \infty$ and let $0 \leq \delta < 2/(m-1)$. Suppose u_ν, $\nu = 1, 2, \cdots$, is a sequence such that $\sum \lambda^{-2-\delta}(u_\nu) < \infty$. Then (iii) $\Phi(y, u) = \mathfrak{I}u^{m/2} + O(u^{(m-1)/4}\lambda(u))$ for almost all y as u runs through the sequence u_ν. The proofs depend on the estimates for integrals and are similar to the corresponding known proofs for ellipsoids [for (i) and (ii) see Landau, S.-B. Preuss. Akad. Wiss. 1915, 458–476; Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1915, 161–171 (1916); and for (iii) see D. G. Kendall, Quart. J. Math., Oxford Ser. (1) 19, 1–26 (1948); these Rev. 9, 570]. An improvement of Minkowski's convex body theorem is also given for bodies B provided that the dimension-number $m \not\equiv 1 \pmod 4$. Independently of the above, a relation is given between the numbers of lattice points in two polar bodies, generalising one of A. Gelfond [C. R. (Doklady) Acad. Sci. URSS (N.S.) 17, 447–449 (1937)] for parallelepipeds. The notation A^* for the transpose of A is used. In equation (3), page 21, read $k = A^{*-1}l$. This definition of k is tacitly assumed in §§ 6–10. There are further minor misprints.

$\qquad\qquad\qquad\qquad$ *J. W. S. Cassels* (Cambridge, England).

Hlawka, Edmund. **Integrale auf konvexen Körpern. II.** Monatsh. Math. **54**, 81–99 (1950).

In the notation of the review of part I [see the preceding review] let $G(l) = \int_B e^{ilx}dx$ and let $H(l)$ be the function of support (Stützfunktion) of B. If B has centre 0 the zeros of $G(l)$ lie with a finite number of exceptions on infinitely many convex surfaces L_k, $k = 1, 2, \cdots$, with the same centre and asymptotically approaching $H(l) = \frac{1}{2}(m+1)\pi + \frac{1}{2}(2k+1)\pi$. If B has no centre, then a ray through 0 has on it only a finite number of zeros of $G(l)$ except in specified cases. If $m = 2$ then $G(l)$ vanishes on infinitely many curves of the type $H(l) = $ constant if and only if B is an ellipse. The estimates of part I for some integrals $\int_B \Phi(f(x))e^{ilx}dx$ are improved. $\qquad$ *J. W. S. Cassels* (Cambridge, England).

Hlawka, Edmund. **Über die Zetafunktion konvexer Körper.** Monatsh. Math. **54**, 100–107 (1950).

Let a convex body B in m-dimensional space be defined by $f(x) \leq 1$. Let the origin O be an interior point of B, and assume that the boundary of B is analytic and that all tangent planes have contact of the first order exactly. Let A be a matrix of determinant 1. The author considers the function $Z(s) = \sum_g'(f(Ag))^{-2s}$, where the summation is over all points g with integral coordinates, excluding 0. This is a special case of ζ-functions considered by Mordell [Quart. J. Math., Oxford Ser. (1) 1, 77–101 (1930)], but reduces to Epstein's ζ-function when B is a sphere with centre O. It is proved that $Z(s)$ is analytic except for a simple pole at $s = \frac{1}{2}m$, with residue $\frac{1}{2}mV(B)$, where $V(B)$ is the volume of B. Other results are given which are partial analogues of results known for Epstein's ζ-function, one being a kind of functional equation, and another being an estimate for the difference between $Z(s)$ and an approximating sum. The proofs use results from the two papers reviewed above.

$\qquad\qquad\qquad\qquad\qquad\qquad$ *H. Davenport* (London).

Yamabe, Hidehiko, and Yujobô, Zuiman. **On the continuous function defined on a sphere.** Osaka Math. J. **2**, 19–22 (1950).

The following theorem is proved for every dimension n. Let $f(X)$ be a real-valued continuous function defined on an n-dimensional sphere S^n in an $(n+1)$-dimensional Euclidean space E^{n+1} with center O. Then there exist $n+1$ points $X_1, \cdots, X_{n+1}$ on S^n such that (i) the vectors $OX_1, \cdots, OX_{n+1}$ are perpendicular to one another and (ii) $f(X_1) = \cdots = f(X_{n+1})$. This theorem was originally conjectured by Rademacher and the case $n = 2$ was proved by the reviewer [Ann. of Math. (2) **43**, 739–741 (1942); these Rev. **4**, 111]. From this follows, as a corollary, that for any bounded convex body in E^{n+1} there exists a circumscribing cube around it. The authors prove this theorem by reducing it to the following lemma which can be proved by induction on n. Let S_0^n, S_1^n be two concentric n-dimensional spheres in E^{n+1} with the same center O. Let L be a closed subset of E^{n+1} contained between S_0^n and S_1^n which intersects any continuous curve joining S_0^n and S_1^n. Then L contains $n+1$ points $X_1, \cdots, X_{n+1}$ such that the vectors $OX_1, \cdots, OX_{n+1}$ are perpendicular to one another and have the same length.

$\qquad\qquad\qquad\qquad$ *S. Kakutani* (New Haven, Conn.).

Radon, Johann. **Über geschlossene Extremalen und eine einfache Herleitung der isoperimetrischen Ungleichungen.** Ann. Mat. Pura Appl. (4) **29**, 315–320; **30**, 309 (1949).

The author's derivation of the isoperimetric inequality is similar to one given by Dinghas [Math. Z. **47**, 677–737 (1942); these Rev. **7**, 528]. The present paper gives a clearer insight into the proof by using standard methods from the calculus of variations. The isoperimetric inequality in the plane can be reduced to the inequality (1) $\lambda L - F \geq \pi \lambda^2$, where L is the length and F the enclosed area of a curve C, which is inscribed in a parallel strip of width 2λ. The semicircles touching the boundaries of the strip form two fields of extremals for the corresponding variational problem of minimizing $\int [\lambda(\dot{x}+\dot{y}^2)^{\frac{1}{2}} - \frac{1}{2}(x\dot{y}-y\dot{x})]dt$. Then (1) is seen to be an immediate consequence of the integral formula of Weierstrass. The same method of proof can be used to derive the corresponding isoperimetric inequalities in Euclidean spaces of higher dimensions and in spaces of constant curvature. $\qquad\qquad$ *F. John* (Los Angeles, Calif.).

Youngs, J. W. T. **The isoperimetric inequality for closed surfaces.** Rivista Mat. Univ. Parma **1**, 189–195 (1950).

The spatial isoperimetric inequality

(1) $$[A(S)]^3 \geq 36\pi[V(S)]^2$$

has been proved recently by Radó [Trans. Amer. Math. Soc. **61**, 530–555 (1947); these Rev. **9**, 137] for all closed Fréchet surfaces S utilizing the Lebesgue definition of area $A(S)$ and a convenient definition $V(S)$ of the volume enclosed by S [see also Tonelli, Rend. Circ. Mat. Palermo **39**, 109–138 (1915)]; $V(S)$ is the integral of the topological index $\gamma(x; S)$ (≥ 0) of each point x of the space with respect to S $(\gamma(x; S) = 0$ on the surface). Let us consider the countable set of all closed surfaces σ obtained by retraction of S for each of the cyclic elements (spheres) of the middle space M of S. Let $\Gamma(x; S) = \sum_\sigma \gamma(x; \sigma)$ and $V'(S)$ be the integral of $\Gamma(x)$. The author proves that (1) $V'(S)$ has the properties of an enclosed volume as well as $V(S)$; (2) $V(S) \leq V'(S)$ for all closed surfaces S and there are surfaces S such that $V(S) < V'(S)$; (3) relation (1) holds also if we substitute $V'(S)$ for $V(S)$. From this fact the author deduces that

for each surface S, for which in (1) equality holds, the middle space M contains only one cyclic element (sphere).

L. Cesari (Lafayette, Ind.).

Algebraic Geometry

Claeys, A. Sur la courbe de Rolle. Mathesis 57, 23–29 (1948).

Hohenberg, Fritz. Eine einfache Fläche achter Ordnung. Monatsh. Math. 54, 140–156 (1950).

La surface Φ que l'auteur considère ici est le lieu des points de l'espace dont les distances d_1, d_2, d_3 à trois points donnés F_1, F_2, F_3, les foyers de Φ, sont liées par la relation: $\pm d_1 \pm d_2 \pm d_3 = 3a$, où a est une constante donnée. L'auteur considère en particulier le cas où F_1, F_2, F_3 sont les sommets d'un triangle équilatère, car la surface qu'on obtient dans ce cas présente toutes les caractéristiques essentielles du cas général: Φ est du 8me degré; elle possède trois circonférences doubles; le cercle absolu de l'espace a pour Φ la multiplicité 4 et équivaut à deux circonférences cuspidales superposées; Φ contient des courbes gauches remarquables du 4me ordre et de la première espèce; etc. En faisant varier a on obtient un système simplement infini de surfaces Φ confocales dont l'enveloppe se compose des plans des trois circonférences doubles et de trois cônes à génératrices isotropes. L'auteur donne l'équation de Φ sous plusieurs formes différentes; la forme suivante est remarquable:

$$\Phi = K_1{}^2 K_2{}^2 + K_2{}^2 K_3{}^2 + K_3{}^2 K_1{}^2 - 2K_0 K_1 K_2 K_3 = 0,$$

où les K_i sont des coordonnées tétrasphériques. Il montre l'analogie entre Φ et la surface de Steiner, dont l'équation s'écrit sous la même forme, mais avec les coordonnées projectives ordinaires. La surface Φ peut être transformée en une surface du 5me ordre par certaines transformations par rayons vecteur réciproques. Enfin l'auteur définit une involution dans l'espace dont Φ est la surface lieu des points doubles.

E. G. Togliatti (Gênes).

d'Orgeval, B. Sur une inégalité de la théorie des surfaces. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 302–304 (1950).

This is a proof by classical algebro-geometrical methods of the inequality $p_g \leq 2p_a + 4$ proved transcendentally by Castelnuovo [Rend. Circ. Mat. Palermo 20, 55–60 (1905)] for an irregular surface not containing an irrational pencil.

P. Du Val (Athens, Ga.).

Dantoni, Giovanni. Sulle singolarità della jacobiana e su quelle della varietà delle ipersuperficie con punto doppio di un generico sistema lineare ∞^r di $V^n{}_{r-1}$ di S_r. Ann. Scuola Norm. Super. Pisa (3) 3 (1949), 1–17 (1950).

Par un choix convenable du système de référence et des formes qui individualisent un système linéaire ∞^r, de $V_n{}^{r-1}$ de S^r non composé avec une involution de variétés de dimension positive, l'auteur établit par l'étude de l'équation de la jacobienne J: (1) Un point P multiple d'ordre s pour la seule variété V_0 qui y soit multiple, est d'ordre $s-1$ pour J, et le cône tangent y est la première polaire par rapport à V_0 de la droite t_0 axe des hyperplans tangents aux V passants en P; si t_0 est génératrice d'ordre s de ce cône, l'ordre de P pour J est supérieur à $s-1$; (2) un point P d'ordre s_1 pour un système d'ordre r_1, d'ordre s_2 pour un sys-

tème d'ordre r_2, $\cdots$, est pour J d'ordre $\sum (r_i - r_{i+1})(s_i - 1)$; (3) un point P double pour un faisceau sans être jamais d'ordre > 2 est point double de J et son cône tangent a un S^{r-4} de points doubles.

Dans l'espace S^N représentatif de toutes les V_n de S^r un compte de constantes montre qu'un système linéaire "générique" ne possède pas de V dotées de points triples ni de points doubles dont le cône tangent ait une génératrice double, tangente à toutes les V qui y passent. Il en résulte que pour un système générique, la J n'a pas de points multiples si $r < 4$ et si $r \geq 4$ elle possède une variété $M_n{}^{r-4}$ de points doubles, correspondant aux points où la matrice $\|\partial f_i/\partial x_j\|$ a la caractéristique inférieure à r, d'où $m = (n-1)^4 r(r+1)^2 (r+2)/12$ [Segre, Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 9_1, 253–260 (1900)].

Dans le même espace S^N, la variété W^{N-1} représentative des V_n dotées d'un point double est rationnelle contenant une involution rationnelle de S^{N-r-1}; deux quelconques d'entre eux définissent un S^{N-2r-2}, formant une involution rationnelle engendrant la $W^{N-2'}$ rationnelle des V_n dotées de deux points doubles; deux S^{N-r-1} infiniment voisins dans l'involution engendrent la $W^{N-2''}$ des V_n dotées d'un point double dont le cône a une génératrice double, variété rationnelle contenant l'involution rationnelle des S^{N-2r-1}. Les points des deux W^{N-2} sont doubles pour la W^{N-1}. Si on coupe cette W^{N-1} par un S^r image d'un système linéaire, la section est birationnellement identique à la J ce qui démontre sa régularité superficielle. *B. d'Orgeval.*

Keller, Ott-Heinrich. Zur Theorie der ebenen, algebraischen Berührungstransformationen. II. Math. Ann. 121, 467–495 (1950).

This paper, a continuation of a paper with the same title [same Ann. 120, 650–675 (1949); these Rev. 10, 736], deals with the singularities of a plane algebraic contact transformation. Let (x_i) and (u_i), $i=1, 2, 3$, be point- and line-homogeneous coordinates in one plane, and (X_i), (U_i) the analogous coordinates in the other plane, and $F(X, x) = 0$ be the "aequatio directrix" of the transformation $(F$, determined but for a factor, can always be supposed rational). The determinant $\Delta = |F_{X_i x_k}|$, $i, k = 1, 2, 3$, is called the "characteristic determinant." Singular places are those pairs of points (X, x) for which $\Delta = 0$. In nonhomogeneous coordinates X, Y, x, y $(X_3 = x_3 = 1)$, considered as Euclidean coordinates in an R_4, the manifolds $F = 0$, $\Delta = 0$ intersect in a surface M. It is proved that the multiplicity of intersection of $F = 0$, $\Delta = 0$ at a point P of M, with all planes having with the tangent plane at P only the point P in common, is the same. This is the multiplicity of intersection of the two manifolds at P. If ρ and σ are two planes having just one point in common, the plane through P parallel to ρ intersects σ at a point, the projection of P on σ parallel to the orientation of ρ. The projection of M on σ reduces to a curve if and only if the projecting plane through every point P intersects the tangent plane at P to M in a line. If ρ and σ have the orientations of the two planes $X = Y = 0$, $x = y = 0$, and the projection of M on one of them parallel to the other is considered, many possibilities arise according as the projection (on one plane or the other or both) is a plane, a curve, or a point. The different cases are examined. Since to an algebraic surface in R_4 three numbers may be attached (numbers of intersection with three different kinds of planes), the formulas giving these numbers for M are three fundamental relations analogous to the two well-known relations in the case of Cremona-transformations. *E. Bompiani.*

Manara, Carlo Felice. Approssimazione delle trasformazioni puntuali regolari mediante trasformazioni cremoniane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 103–108 (1950).

The possibility of approximating an analytic transformation between two projective planes in any neighborhood of two corresponding regular points is proved by examining the elementary transformations of the following type: $x' = x + ax''y^s$, $y' = y$, $r \geqq 0$, $s \geqq 0$. These may always be approximated by products of de Jonquières transformations. By a recurrence procedure the theorem is extended to any two projective spaces (of the same dimension).

                                        E. Bompiani (Rome).

Villa, Mario, e Vaona, Guido. Sul caso cremoniano delle trasformazioni puntuali fra due piani o spazi. Boll. Un. Mat. Ital. (3) **5**, 48–54 (1950).

Considering a point-to-point transformation between two projective planes (or spaces) and a pair of corresponding points (O, O') where the Jacobian is zero (O a simple point of the Jacobian), (1) it is possible to approximate the given transformation up to and including the neighborhood of the second order of (O, O') by a Cremona transformation if and only if the curves (surfaces) corresponding to the lines (planes) through O' have second order contact, (2) in the case of two spaces the last condition is equivalent to the condition that the stationary line through O is tangent to the Jacobian surface.

                                        E. Bompiani (Rome).

Segre, Beniamino. Corrispondenze analitiche e trasformazioni cremoniane. Ann. Mat. Pura Appl. (4) **28**, 107–139 (1949).

This paper concerns the possibility of approximating by Cremona transformations a given analytic point-to-point transformation between two projective planes up to and including the neighborhood of a given order n of a regular pair of corresponding points (O, O'). This problem was posed by the reviewer and solved by Villa [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 718–724 (1942); **4**, 1–7 (1943); these Rev. **8**, 219] and the reviewer [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **13**, 837–848 (1942); these Rev. **8**, 219] for $n = 2$. The main results of the present paper are the following theorems. (1) The product of $\mu_n = [\frac{1}{4}(n^2 + 5n - 16)]$ (or more) quadratic generic transformations is a Cremona transformation which behaves like a general transformation up to and including the neighborhood of order n of two of its corresponding generic points. (2) An analytic transformation may be approximated up to and including the neighborhoods of order n of any given pair of corresponding points by a Cremona transformation which is the product of at most $\nu_n = \frac{1}{4}(n^4 + 10n^3 - 9n^2 - 82n + 108)$ quadratic transformations. The problem of finding the Cremona transformation of minimum order approximating to order n a given transformation is also tackled and solved for $n = 3$. Results are also given concerning properties, both local and in the large, of Cremona transformations.

                                        E. Bompiani (Rome).

Segre, Beniamino. Corrispondenze analitiche e trasformazioni cremoniane. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 49–55 (1949).

Summary of the paper reviewed above.

✶**Segre, Beniamino.** Problèmes arithmétiques en géométrie algébrique. Colloque de géométrie algébrique, Liège, 1949, pp. 123–142. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

The first part of this paper is an incomplete discussion of a problem completely solved by the author [Boll. Un. Mat. Ital. (3) **5**, 33–43 (1950); these Rev. **12**, 50]. The second part discusses quadratic forms in a field γ of arbitrary characteristic. Many of the results were already obtained by Witt [J. Reine Angew. Math. **176**, 31–44 (1936)] but with the restriction that the characteristic of γ is not 2. If f', f'' are forms, write $f' \overset{\gamma}{\rightarrow} f''$, $f' \overset{\gamma}{\leftrightarrow} f''$ for "f' represents f'' in γ" and "f' is equivalent to f'' in γ," respectively. If $f(x_1, \cdots, x_n) \overset{\gamma}{\leftrightarrow} f'(y_1, \cdots, y_n)$, where for some $r > 0$ the coefficients of all terms involving $y_1, \cdots, y_r$ are zero, then f is called singular, otherwise nonsingular. Let $f_1' = f' + \bar{f}'$; $f_1'' = f'' + \bar{f}''$, where f', $\bar{f}'$, f'', $\bar{f}''$ are nonsingular forms in different variables and f_1', f_1'' are also nonsingular. Then

(a) $f_1' \overset{\gamma}{\rightarrow} f_1''$ and $f' \overset{\gamma}{\leftrightarrow} f''$ imply $\bar{f}' \overset{\gamma}{\rightarrow} \bar{f}''$

(b) $f_1' \overset{\gamma}{\leftrightarrow} f_1''$ and $f' \overset{\gamma}{\rightarrow} f''$ imply $\bar{f}'' \overset{\gamma}{\rightarrow} \bar{f}'$

(c) $f_1' \overset{\gamma}{\leftrightarrow} f_1''$ and $f' \overset{\gamma}{\leftrightarrow} f''$ imply $\bar{f}' \overset{\gamma}{\leftrightarrow} \bar{f}''$.

Further, every $f(x_1, \cdots, x_n)$ is equivalent in γ to a form $y_1 y_1' + \cdots + y_h y_h' + \phi(z_1, \cdots, z_k)$ where ϕ does not represent 0 and $2h + k \leqq n$, with equality if and only if f is nonsingular. Here h, k are invariants of f, and ϕ is unique up to equivalence. Further, if $f \overset{\gamma}{\rightarrow} f'$ then $h \geqq h'$, $h + k \geqq h' + k'$ (h, k correspond to Sylvester's index of inertia). Proofs depend on the invariant characterisation of h, k in terms of the geometry of the "quadric surface" $f(x_1, \cdots, x_n) = 0$ ($x_1, \cdots, x_n$ in γ). Further results are indicated, some without proof.

The third part of the paper considers two varieties U, V defined in γ and birationally equivalent in some field over γ and asks: (1) Is U birationally equivalent to V in γ? (2) If not, in what fields over γ is U equivalent to V? (3) Can we construct in γ a variety W, connected with U in some simple way and birationally equivalent to V in γ? It is shown that problem (3) can be solved subject to certain conditions if U is a hypersurface (so that V is a "generalised Severi-Brauer variety") and that this gives an approach to (1) and (2).   *J. W. S. Cassels* (Cambridge, England).

Errera, A. Un problème diophantien de M. Segre. Bull. Soc. Roy. Sci. Liège **19**, 177–186 (1950).
Errera, A. Un problème diophantien de M. Segre: Addenda. Bull. Soc. Roy. Sci. Liège **19**, 213–214 (1950).

It is shown that the only rational solutions x, y of $y^2 = 3x(x^2 + x + 1)$ are the trivial ones $x = y = 0$, $x = 1$, $y = \pm 3$. This solves a problem of Segre [see the preceding review; for another solution see Segre, Boll. Un. Mat. Ital. (3) **5**, 33–43 (1950); these Rev. **12**, 50]. It is also shown that there are no integer solutions x, y of $y^2 = p^u x(x^2 + x + 1)$, $u \geqq 0$, p prime, except $x = y = 0$ and $p = 3$, $u = 1$ (2), $x = 1$.

                                        J. W. S. Cassels (Cambridge, England).

Barsotti, I. Algebraic correspondences between algebraic varieties. Ann. of Math. (2) **52**, 427–464 (1950).

This paper gives a general theory of correspondences between algebraic varieties, and from this develops the theory of algebraic systems of cycles on an algebraic variety. The varieties considered are defined by ideal-theoretic means, the ground field being quite arbitrary. After giving precise

definitions and some preliminary algebraic results, the notion of the general projection of a cycle is introduced. This notion bears a relationship to the associated form of Chow and van der Waerden and, among other things, leads to a convenient definition of the degree of a cycle. The concept of general projection is one of the essential tools of the theory which follows.

An algebraic correspondence between the varieties F and V is defined by means of a cycle D on the direct product $F \times V$. If P is a point of F, W a sub-variety of V, W is a correspondent of P: $W = D(P)$, if there exists a prime ideal of the ring of $F \times V$ which contains the radical of the ideal of D whose projection on F is the ideal of P in the ring of F and whose projection on V is the ideal of W in the ring of V. From this definition, the author passes to the idea of a correspondence between a function field K (the function field of F) and V, the correspondence being between the zero-dimensional valuation of K and sub-varieties of V. This enables him to develop the properties of correspondences in detail, particularly in regard to theorems on the dimension of the total transform on V of a point on F, and on the fundamental locus on F. To give a clear statement of the results obtained would necessitate the quotation of lengthy passages from the paper.

An algebraic system of cycles on a variety V of r dimensions is a set S of cycles having the property that there exists a field K and an unmixed algebraic correspondence Δ between K and V such that S consists of the set of cycles of V corresponding to the zero-dimensional valuations of K. Leaning on the properties of algebraic correspondences already developed, the author develops the theory of algebraic systems on a variety, introducing the notion of irreducible, prime, and simple systems. He establishes properties of systems compounded from other systems, of involutions, and of the index of systems, and concludes with a study of pencils of $(r-1)$-dimensional cycles on V, including the theorem that if a pencil on an irreducible variety has a simple point of the variety as a base point, it is simple. The author states that a list of corrections is forthcoming.

<div align="center">W. V. D. Hodge (Cambridge, England).</div>

Severi, Francesco. Fondamenti per una teoria generale dei connessi. Acta Salmanticensia. Ciencias: Sec. Mat., no. 3, 28 pp. (1950).

In the terminology of Clebsch a subvariety of the Cartesian product of a finite number of varieties is called a connex or connexion. The present paper gives a geometric theory of unmixed algebraic connexions of maximum dimension on the product $[r_1] \times \cdots \times [r_q]$ of q complex projective spaces. This is equivalent (birationally) to a theory of unmixed subvarieties W_{t-1} of the variety V_t ($t = \sum r_i$) of Segre in $[\rho]$ ($\rho = \prod(r_i+1)-1$) associated with $[r_1] \times \cdots \times [r_q]$. The main results are deduced from the following theorem which is proved by induction on q: If $x_0{}^i, x_1{}^i, \cdots, x_r{}^i$ are homogeneous coordinates of the general point of $[r_i]$, then every unmixed W_{t-1} can be represented by a multiply homogeneous form $f(x^1; \cdots; x^q) = 0$. If f is homogeneous of degree n_i in the variables (x^i), the integers $n_1, \cdots, n_q$ are called the indices of the connex. The theorem does not imply that W_{t-1} as a subvariety of V_t will be the complete intersection of V_t with a hypersurface of $[\rho]$. A necessary and sufficient condition for this to be so is that $n_1 = \cdots = n_q$. The author seems to have overlooked the fact that these two results were proved by Goddard [Proc. Cambridge Philos. Soc. 39, 35–48 (1943); these Rev. 4, 168].

The theory of linear systems of connexes W_{t-1} on V_t is discussed briefly, and it is shown that if H_j is a hyperplane in $[r_j]$ and if $I_j = [r_1] \times \cdots \times [r_{j-1}] \times H_j \times [r_{j+1}] \times \cdots \times [r_q]$, then $I_1, \cdots, I_q$ form a base for the W_{t-1} on V_t. A W_{t-1} of indices n_i is linearly equivalent to $n_1 I_1 + \cdots + n_q I_q$. [This result was also obtained by Goddard.] With the aid of this base a Bézout theorem for the intersection of t connexes W_{t-1} on V_t of arbitrary indices is formulated and proved. It is also shown that the variety of Segre, V_t, is arithmetically normal in the sense of Zariski in its ambient space $[\rho]$.   *H. T. Muhly* (Iowa City, Iowa).

Severi, Francesco. Sulle molteplicità d'intersezione delle varietà algebriche ed analitiche e sopra una teoria geometrica dell'eliminazione. Math. Z. **52,** 827–851 (1950).

The definition of intersection multiplicity, which was formulated by the author [Abh. Math. Sem. Hamburg. Univ. 9, 335–364 (1933)], is restated as follows: If the subvarieties V_h and W_k of an irreducible variety M_r ($h+k=r$) have the simple point P of M_r as an isolated common point, then P is said to be a simple intersection of V and W or an intersection of multiplicity one if V and W have no common tangents at P. In general when the simple point P is an isolated common point of V_h and W_{r-h}, it is possible to select (in infinitely many ways) a variety $V_h{}'$ not on P such that the variety $H = V + V'$ belongs to an irreducible system Σ on M_r, such that the general variety H^* of Σ in the neighborhood of H has a certain number i of simple intersections with W. (If V itself belongs to such a system the introduction of the auxiliary V' is not necessary.) This number i is the intersection multiplicity of V and W at P. It is not changed if the rôles of V and W are reversed. When $h+k>r$ the problem of the multiplicity of intersection at the general point of an irreducible component C of $V \cap W$ of dimension $t = h+k-r$ is reduced to the former case by taking a section of the configuration by a general linear S_{d-t} of the ambient space S_d of M_r.

The definition given by A. Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. 9, 303] is compared to the one given in the present paper. (The author points out that he had neither the time nor the opportunity to consult the work of Chevalley [Trans. Amer. Math. Soc. 57, 1–85 (1945); these Rev. 7, 26] on the same subject.) For varieties defined over the complex number field, the equivalence of the present definition (S) with that of Weil (W) is demonstrated by showing first that if U_0 is the diagonal of the product $M_r \times M_r'$ (M_r' is another copy of M_r), then the multiplicity (S) of intersection of V_h and W_{r-h} at an isolated common point P (simple for M_r) is equal to the multiplicity (S) of intersection of $V_h \times W_{r-h}$ with U_0 at $P \times P$. It is then shown that if S_ρ ($\rho = d(d+2)$) is the ambient space of $S_d \times S_d'$, then the latter multiplicity is equal to the multiplicity of intersection at $P \times P$ of $V \times W$ with a generic linear $S_{\rho-r}$ passing through U_0. It is pointed out that definition (S) is intrinsic (to M_r) and invariant under analytic transformations of M_r which are biregular at P. The remark is made that while definition (W) has the same invariant properties it is not intrinsic. The extension of definition (S) to the case of algebroid varieties on a linear r-dimensional branch of origin P is also discussed.

The paper includes a geometrical proof of the Bézout theorem for r hypersurfaces in S_r, and a discussion of a geometrical method for obtaining the resultant of $r+1$ forms in $r+1$ variables and the u-resultant of r forms.

<div align="center">H. T. Muhly (Iowa City, Iowa).</div>

✱Samuel, P. **Multiplicités des composantes singulières d'intersection.** Colloque de géométrie algébrique, Liège, 1949, pp. 87–90. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

The author extends the notion of the intersection multiplicity of two cycles X and Y on a variety A to certain cases in which a component M of the intersection is singular on A. In order to do so, he has to assume that X and Y are rational cycles which are locally, at M, complete intersections of A with rational cycles of the ambient space (or, which amounts to the same, that X and Y are integral cycles which are locally submultiples of complete intersections of A with cycles of the ambient space). In that case, an intersection multiplicity is defined which has the usual properties relative to associativity, projections, and products; this multiplicity may fail to be integral even if X and Y are integral (the multiplicity of the vertex of a quadratic cone in the intersection of two generators of the cone is $\frac{1}{2}$).

The author then defines a new invariant of the pair formed by a variety A and a subvariety N of A, which may be singular. First, the local cycles at N are defined to be the cosets of the group of cycles modulo the subgroup of those cycles no component of which contains N; then the holotomy group at N is defined to be the factor group of the group of local cycles at N by the complete intersections of A with cycles of the ambient space. If A is a cone of vertex M, then every local divisor at M is shown to be holotomic to a conic divisor; this permits explicit determination of the holotomy group in certain cases.

The following results are stated: (a) If A is normal at M, a positive local divisor at M which is holotomic to 0 is representable as the intersection of A with a positive cycle of the ambient space; (b) if A is normal at M, then a necessary and sufficient condition for every divisor at M to be holotomic to 0 is that the quotient ring of M on A be a unique factorization ring. This may happen even when M is singular, which gives an example of a local ring in which unique factorization holds but which is not regular.

C. Chevalley (New York, N. Y.).

✱Dubreil-Jacotin, M.-L., et Dubreil, P. **Divers types d'anneaux intervenant en géométrie algébrique.** Colloque de géométrie algébrique, Liège, 1949, pp. 57–78. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

An irreducible variety in projective n-space over a field K is "arithmetically normal" (a.n.) if its homogeneous coordinate ring P is integrally closed in its field of quotients F or, equivalently (since P is Noetherian), if P is "complètement entier-fermé" (c.e.f.), i.e., if $a\varepsilon F$, $0 \neq r\varepsilon P$, $ra^m\varepsilon P$ $(1 \leq m < \infty)$ imply $a\varepsilon P$ (in general, for integral domains c.e.f. implies integral closure but not conversely). If $\mathfrak{A}$ is an ideal in $K[X_0, \cdots, X_n]$ without improper component (i.e., without primary component belonging to $(X_0, \cdots, X_n)$) and if Φ is a form not in any prime component of $\mathfrak{A}$, then the condition that $\mathfrak{A}+(\Phi)$ be without improper component is independent of Φ; if $\mathfrak{A}$ satisfies this condition, $\mathfrak{A}$ is of the "first kind." A variety is of the first kind if its radical ideal is. The study of these two concepts and of certain known results in algebraic geometry leads the authors to try to extend the notion of "a.n." to not necessarily irreducible varieties. Broadly the plan is as follows. The study of a variety V is replaced by that of its homogeneous coordinate ring P; P is then regarded as an abstract commutative ring A with unity (possibly with zero divisors). The authors form the ring A_S of quotients of A with respect to any suitable subset S, and in A_S consider certain ideals (regular fractional ideals); the set of all these is a lattice T, and the study of A is replaced by that of an abstract lattice having certain properties abstracted from T (residuated multiplicative lattice). After extending the definition of "a.n." to arbitrary varieties (in terms of certain types of lattice equivalence relations), the authors prove the theorem: If V is a.n. and has no component of dimension <1, then V is of the first kind. E. R. Kolchin (New York, N. Y.).

✱van der Waerden, B. L. **Les variétés de chaînes sur une variété abstraite.** Colloque de géométrie algébrique, Liège, 1949, pp. 79–85. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

A large part of the paper consists in the exposition of the definition of the "associated form" of a cycle in projective space and of the notion of abstract Variety of A. Weil. The author proposes two terminological innovations: "indivisible variety" for the absolutely irreducible varieties; and "chain" for the cycles on a Variety. Then the author indicates how it is possible to define the notion of specialisation for cycles on a Variety. In a concluding remark, the author states that the natural objects to be considered on a variety of dimension d, inasmuch as one is concerned with birationally invariant properties, are the $(d-1)$-dimensional valuations. However, it seems to the reviewer that the work of Zariski [in particular, his proof of the theorem of local-uniformisation, Ann. of Math. (2) **41**, 852–896 (1940); these Rev. 2, 124] shows rather conclusively that the consideration of valuations of dimension $d-1$ only is not sufficient: valuations of lower dimension, and in particular of dimension 0, have to be considered too. C. Chevalley.

Differential Geometry

Mirguet, Jean. **Surfaces dont l'ensemble des points à double courbure est dense.** C. R. Acad. Sci. Paris **231**, 24–26 (1950).

L'auteur, dans des articles antérieurs [Revue Sci. **85**, 67–72 (1947); **86**, 323–328 (1948); mêmes C. R. **228**, 646–648, 1474–1476 (1949); ces Rev. 9, 18; 10, 567, 621], a étudié des propriétes du second ordre de surfaces du point de vue de la géométrie infinitésimale de Bouligand. Les surfaces étudiées sont des "orthosurfaces" (éléments lipschitziens) S à paratingent second fini. Désignant par Δ une direction de droite exclue des paratingents ordinaires aux divers points x de S, la fixation d'une orientation sur Δ permet d'attribuer un signe aux éléments linéaires tangents (x, t), sauf peut-être en un nombre fini d'entre eux pour chaque x, ce signe indiquant la position locale en x de l'intersection de S avec le demi-plan défini par t et le rayon issu de x parallèle à la direction positive de Δ; un point x est pourvu d'un signe dit superficiel lorsque les signes des éléments linéaires tangents issus du point sont les mêmes à l'exception d'un nombre fini d'entre eux. La surface est dite "à double courbure" en x lorsque deux infinités de demi-tangentes orientées t ont des signes contraires. Chacune des hypothèses suivantes entraîne la densité topologique sur S de l'ensemble des points à double courbure: Tout voisinage sur S contient des points de signes superficiels contraires; S n'admet pas de plans d'appui locaux. Le paratingent

second comprend au moins une droite et le paratingent quatrième est vide. La surface S peut être engendrée simultanément par des arcs positifs et négatifs, dans des plans verticaux.   *C. Pauc* (Le Cap).

Sanielevici, S.   Remarque sur les podaires négatives d'une courbe gauche.   Acad. Repub. Pop. Romậne. Bul. Şti. A. 1, 369–374 (1949).   (Romanian, Russian, and French)

Garnier, René.   Extension d'une formule de Lie aux espaces cayleyens.   Bull. Soc. Roy. Sci. Liège 19, 187–191 (1950).
Let $c_{ij} = -c_{ji}$ be the Plücker coordinates of a complex L in an elliptic three space and $x^i = x^i(s)$ the Weierstrass coordinates of points of a curve C. Put $[p, q] = c_{ij}p^i q^j$ for arbitrary p, q. If C belongs to L, then (1) $[xx'] = 0$, and consequently also (2) $[xx''] = 0$, and (3) $[xx'''] + [x'x''] = 0$. Using the equations (1), (2), (3), the Frenet formulas for C, and a special coordinate system, one obtains easily the generalization of the classical Lie result for the torsion of C.
                               V. Hlavatý (Bloomington, Ind.).

Mandan, Ram.   Umbilical projection in four dimensional space S_4.   Proc. Indian Acad. Sci., Sect. A. 28, 166–172 (1948).

Bhattacharya, P. B., and Behari, Ram.   Some properties of the skewness of distribution of the generators of a ruled surface.   Bull. Calcutta Math. Soc. 42, 37–42 (1950).
V. Ranga Chariar [même Bull. 37, 133–136 (1945); ces Rev. 7, 392] a défini un nombre μ (skewness) caractérisant la distribution des génératrices d'une surface réglée au voisinage d'une génératrice donnée. Les auteurs donnent quelques nouvelles propriétés de ce nombre μ. Ils montrent d'abord que pour une développable $1/\rho_1 = u/\mu$, $1/\rho_1$ étant la courbure principale non nulle en un point quelconque, et u la distance du point à l'arête de rebroussement comptée sur la génératrice envisagée. Pour la surface réglée formée par les normales principales d'une courbe gauche $\mu = \sigma/\rho$, ρ et σ désignant les rayons de courbure et de torsion de la courbe gauche au point d'où est issue la normale principale que l'on considère. Pour la surface réglée formée par les binormales d'une courbe gauche, on a $\mu = \rho\sigma(\rho'\sigma - \sigma'\rho)/(\rho^2+\sigma^2)^{\frac{1}{2}}$, formule qui prouve que si la courbe est une hélice $\mu = 0$. L'introduction des asymptotiques curvilignes d'une surface réglée montre que ces courbes peuvent être obtenues par deux quadratures si, pour la surface réglée, $\mu = 0$. Les auteurs recherchent ensuite l'expression de μ pour les surfaces réglées issues d'un rayon d'une congruence rectiligne. Ils examinent en détail le cas des cinq familles de surfaces réglées remarquables contenues dans toute congruence rectiligne, et montrent, en particulier, que si la congruence est normale μ est le même pour les développables et les surfaces réglées principales, l'identité de μ pour ces deux familles de surfaces réglées d'une congruence rectiligne étant d'ailleurs une condition nécessaire et suffisante d'orthogonalité de la congruence.   *P. Vincensini* (Marseille).

Mishra, R. S.   A note on parameter of distribution of a ruled surface through a line of a rectilinear congruence.   Bull. Calcutta Math. Soc. 42, 53–56 (1950).
L'auteur donne une nouvelle expression du paramètre de distribution, le long d'une génératrice, d'une surface réglée appartenant à une congruence rectiligne. Si d est ce paramètre, et si t désigne l'abscisse du point central de la génératrice envisagée comptée à partir d'une surface de

référence fixe, la somme $d^2 + t^2$ est mise sous la forme du quotient de deux formes différentielles quadratiques par rapport aux différentielles des deux variables u^i ($i = 1, 2$) qui définissent les différents rayons de la congruence considérée. L'annulation de d ou de t fournit des expressions pour l'abscisse du point central d'une développable de la congruence et pour les paramètres de distribution des surfaces réglées dont les lignes de striction sont sur la surface de référence. Application de ces expressions est faite aux congruences de Ribaucour et aux congruences de ce type qui sont en outre isotropes.   *P. Vincensini* (Marseille).

Mathéev, A.   Sur la géométrie différentielle des surfaces réglées de l'espace elliptique.   Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44, 235–308 (1948).   (Bulgarian.   French summary)

Petkantschin, B.   Allgemeine isotrope Regelscharen im Euklidischen Raum.   Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44, 357–399 (1948).   (Bulgarian.   German summary)

Rozet, O., et Paquet, H.   Sur une classe de surfaces à lignes de courbure planes dans les deux systèmes.   Bull. Soc. Roy. Sci. Liége 18, 343–346 (1949).
Let a surface S be described by a point x, the parametric curves on S being its lines of curvature. This paper studies surfaces S for which the lines of curvature are all plane curves. Such surfaces depend on two arbitrary functions $U(u)$, $V(v)$ of the curvilinear coordinates on S and on an arbitrary constant a. Among such surfaces S are Weingarten surfaces (that is, surfaces for which the congruence of normals are Weingarten congruences). Such Weingarten surfaces are determined. These surfaces depend on two arbitrary constants. For these surfaces the sum of the principal radii of normal curvature is a constant. The minimal surfaces S with plane lines of curvature are determined. They turn out to be the minimal surfaces of Bonnet.
                               V. G. Grove (East Lansing, Mich.).

Rozet, O.   Sur les surfaces à lignes de courbure sphériques.   Bull. Soc. Roy. Sci. Liège 19, 15–17 (1950).
Conditions on the principal radii of normal curvature are found which are necessary and sufficient that the lines of curvature be spherical, the lines of curvature being parametric.   *V. G. Grove* (East Lansing, Mich.).

Rozet, O.   Sur les surfaces de Klein-Lie et sur leurs transformées de Lie.   Bull. Soc. Roy. Sci. Liége 18, 395–398 (1949).
A surface S of Klein-Lie has equations of the form $x^i = \exp A_{ij}U^j$, where $A_{i1} = m_i - m_4$, $A_{i2} = m_i^2 - m_4^2$, and m_1, m_2, m_3, m_4 are the roots of the quartic $(m^2-h)^2 - m - k = 0$. For $h = k = 0$, the surface has the nonparametric equation $xyz = 1$. The asymptotic curves on the surface are parametric. The surface S is transformed into a surface $\bar{S}$ by the transformation of Lie. Under this transformation the lines of curvature on $\bar{S}$ are parametric, i.e., the asymptotic curves on S correspond to the lines of curvature on $\bar{S}$ in the transformation of Lie. The quadric of Lie of S transforms into cycloids of Lie of $\bar{S}$. Hence theorems connecting these quadrics and cycloids and their respective surfaces S and $\bar{S}$ may be found.   *V. G. Grove* (East Lansing, Mich.).

Inzinger, Rudolf. Über eine projektive Invariante eines Paares von Flächenelementen zweiter Ordnung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 157, 263–274 (1950).

Let F_2, F_2' be two surfaces in a projective space S_3 having at their points P, P' ($P \neq P'$) the same tangent plane π (the line PP' is not an asymptotic tangent at P or P'). The second order caps of F_2, F_2' with centers at P, P' have a projective invariant which is the square of the invariant (cross-ratio) of the perspective collineations changing one cap into the other. In case the space is a Euclidean one, metric expressions are given for this invariant. Similar projective and metric interpretations of the invariants of two caps, also in more general conditions, have already been given by the reviewer [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 184–190 (1945); these Rev. 9, 60; other references can be found in the cited paper].

E. Bompiani (Rome).

Backes, F. Sur l'existence d'une surface d'après les deux formes quadratiques fondamentales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 251–256 (1950).

The author gives a variation of Bianchi's proof of Bonnet's theorem on the existence of a surface having given functions $g_{\alpha\beta}$, $d_{\alpha\beta}$ as the covariant components of its fundamental tensors. The method uses completely integrable systems of partial equations whose solutions are known to exist by methods of successive approximation. The existence of a set of functions $x^i = x^i(u^1, u^2)$, and N^i, such that at $u^1 = u_0^1$, $u^2 = u_0^2$,

$$\delta_{ij} \frac{\partial x^i(0)}{\partial u^\alpha} \frac{\partial x^j(0)}{\partial u^\beta} = g_{\alpha\beta}^{(0)}, \quad \delta_{ij} \frac{\partial x^i(0)}{\partial u^\alpha} N^j(0) = 0,$$

$$\delta_{ij} N^i(0) N^j(0) = 1, \quad g_{\alpha\beta}^{(0)} = g_{\alpha\beta}(u_0^1, u_0^2), \text{ etc.}$$

is shown; i.e., there is a surface with equations $x^i = x^i(u^1, u^2)$ which at (u_0^1, u_0^2) has the functional values of $g_{\alpha\beta}^{(0)}, d_{\alpha\beta}^{(0)}$ for the components of its fundamental tensors at the point. A completely integrable system of equations is shown to exist having the six functions $(\delta_{ij}(\partial x^i/\partial u^\alpha)(\partial x^j/\partial u^\beta))$, $(\delta_{ij}(\partial x^i/\partial u^\alpha)N^j)$, $(\delta_{ij}N^iN^j)$ as unique solutions at $u^1 = u_0^1$, $u^2 = u_0^2$. These equations are satisfied by $g_{\alpha\beta}$, 0, 1 for all u^1, u^2. The proof is therefore complete.   *V. G. Grove.*

Verbickiĭ, L. L. On the metric differential geometry of hypersurfaces of the second order. Trudy Sem. Vektor. Tenzor. Analizu 7, 319–340 (1949). (Russian)

A report on the main results of this paper appeared previously [Doklady Akad. Nauk SSSR (N.S.) 60, 1117–1118 (1948); these Rev. 10, 67]. A hypersurface V_{n-1} in a Euclidean E_n is introduced by the radius vector $\vec{r} = \vec{r}(u^i)$, $i = 1, 2, \cdots, n-1$. Then the first and second fundamental tensors are given by $g_{ij}du^i du^j$, $\pi_{ij}du^i du^j$. With the aid of these tensors the so-called tensor of Darboux is introduced:

$$\vartheta_{ijk} = \pi_{ij|k} - [1/(n+1)k](k_i\pi_{jk} + k_j\pi_{ki} + k_k\pi_{ij}),$$

where $\pi_{ij|k}$ is the covariant derivative and k is the product of the principal curvatures, supposedly unequal to 0. When $\vec{\pi}^{ij}\pi_{jk} = \delta_k^i$, then $\vec{\pi}^{ij}\vartheta_{ijk} = 0$. The tensor ϑ_{ijk} is only changed by a multiplicative factor if the V_{n-1} is transformed by a projective transformation of the Cartesian coordinates of the E_n. This shows that $\vartheta_{ijk} = 0$ for every nondegenerate hyperquadric, since $\vartheta_{ijk} = 0$ for a hypersphere. The author then shows that $\vartheta_{ijk} = 0$ also characterizes a hyperquadric. It is sufficient to prove this for the case when the lines of curvature are uniquely determined. The author first shows

that $\vartheta_{ijk} = 0$ means that the V_{n-1} is normal, which means that every congruence of lines of curvature admits a family of orthogonal V_{n-2} in the V_{n-1}. From this he obtains the following five theorems. (I) If a hyperquadric in the E_n has uniquely determined lines of curvature, then it is normal, and the orthogonal V_{n-2} are level surfaces of $\sigma^{n+1}k^{-1}$, where σ is the corresponding principal curvature. (II) If for a V_{n-1} in the E_n the function $\sigma^{n+1}k^{-1}$ is constant on V_{n-2} normal to the corresponding congruence of lines of curvature, then it is a hyperquadric. (III) Along a line of curvature of a surface of the second order the corresponding principal curvature is proportional to the cube of every other principal curvature (σ_i/σ_j^3 is either constant or depends only on u^i, $i \neq j$). (IV) A normal V_{n-1} with constant σ_i/σ_j^3 along the lines of curvature is a hyperquadric. (V) On a hyperquadric of the E_n the ratio $\sigma^{n+1}k^{-3}$ is constant along a given geodesic, where σ is its curvature. Then, by actually integrating the Gauss-Codazzi equations under the condition that $\vartheta_{ijk} = 0$, it is shown that the expressions obtained for g_{ij} can be realized on a surface of the second order $a_1x_1^2 + \cdots + a_nx_n^2 = 1$ with all a_i different from each other. It has thus been shown that $\vartheta_{ijk} = 0$ characterizes the hyperquadrics. [In Hlavatý's Differentialgeometrie . . . [Noordhoff, Groningen-Batavia, 1939, pp. 435 ff.; these Rev. 1, 27], the theorem is proved for the case of ruled surfaces in E_3.]

D. J. Struik (Cambridge, Mass.).

Sauer, Robert. Parallelogrammgitter als Modelle pseudosphärischer Flächen. Math. Z. 52, 611–622 (1950).

It is the purpose of the author to point out some of the analogies between theorems concerning pseudospheres and a certain type of lattice, the meshes of which are composed of skew parallelograms. Let A_0 be an interior point of the lattice. The sides of all parallelograms are S', S''. The nonadjacent vertices (to A_0) of any two parallelograms with common vertex A_0 are constrained to lie in a plane, the so-called tangent plane to the lattice at A_0. The curvature angles k', k'' at A_0 are the angles at A_0 between the sides of equal length S', S'', respectively, of any two distinct parallelograms with vertex A_0. The torsion angles w', w'' are the dihedral angles with edges of length S', S'', respectively, formed by the planes determined by the vertices of a parallelogram with vertex A_0 and the sides of those lengths. The curvatures κ', κ'' and torsions τ', τ'' are defined by the formulas $S'\kappa' = \sin k'$, $S''\kappa'' = \sin k''$, $S'\tau' = \sin w'$, $S''\tau'' = \sin w''$. The osculating planes of a mesh of the lattice are the planes determined by adjacent sides of the mesh. The definition of a tangent plane to a lattice at a vertex gives a definition of a normal to the lattice, and thence a method of constructing a spherical representation of the lattice. The total curvature of a lattice at a point is defined in terms of the area of the spherical representation of a mesh of the lattice and area of the tetrahedron whose vertices are the vertices of the mesh. Using these notions the following sample analogies may be quoted. The tangent planes of a plane lattice coincide with the osculating planes of the meshes. The corresponding statement concerning surfaces is the familiar one about the osculating planes of the asymptotic curves coinciding with the tangent planes. Next, if one of the torsions of a lattice τ' is constant, the other is also constant and $\tau' = -\tau''$. Analogously, if the asymptotic curves form a net of Tchebychef, and if one of the family of asymptotic curves has constant torsion, so also has the other, and consequently the surface is a pseudosphere. Deformations of a lattice preserving the length of sides S', S'' are considered,

and the corresponding theorems concerning the deformations of pseudospheres preserving the lengths and curvatures of the asymptotic curves. Another type of deformation of a lattice preserves the torsion angles and the angles of the parallelograms of the meshes of the lattice. The corresponding theorems concerning surfaces is to the effect that a pseudosphere may be deformed into a pseudosphere with preservation of the angles between the asymptotic curves, the total curvature of the given pseudosphere being preserved by the deformation (the deformation of Lie). Finally, it is shown that by a certain construction of lattice and by a type of limiting process the osculating planes, the curvatures, torsions, and the total curvature of the lattice converge to the osculating planes, the curvatures and torsions of the asymptotic curves of a pseudosphere, and to the curvature of that surface. *V. G. Grove* (East Lansing, Mich.).

Lemoine, Simone. Sur l'emploi d'un repère canonique dans l'étude des surfaces isométriques d'une surface donnée avec correspondance d'un réseau conjugué. C. R. Acad. Sci. Paris **230**, 1571–1573 (1950).

This paper concerns itself with the problem of finding the second fundamental tensor of a surface, having given the metric tensor and a conjugate net on the surface. An application is made to the deformation of such a surface, the given conjugate net being deformed into a conjugate net. The method used is that of moving forms of reference of Cartan. *V. G. Grove* (East Lansing, Mich.).

Samelson, Hans. On Chern's invariant for Riemannian 4-manifolds. Proc. Amer. Math. Soc. **1**, 415–417 (1950). This paper is concerned with the complex projective plane endowed with its elliptic metric. For this space the author verifies the generalized Gauss-Bonnet formula and finds that the Chern invariant [Bull. Amer. Math. Soc. **51**, 964–971 (1945); these Rev. **7**, 216] η has the value -6. This is the first known example of a space for which $\eta \neq 0$.
C. B. Allendoerfer (Haverford, Pa.).

Walker, A. G. Canonical forms. II. Parallel partially null planes. Quart. J. Math., Oxford Ser. (2) **1**, 147–152 (1950).

This is a continuation of an earlier paper by the author [same vol., 69–79 (1950); these Rev. **11**, 688], in which he obtained a canonical form for the metric of a Riemannian n-space that admits a null parallel r-plane (parallel field of r-dimensional vector spaces, all vectors null). In the present paper he considers the same problem for a partially null parallel $(r+s)$-plane of nullity r, which for the purposes of this abstract may be defined as a parallel field of vector spaces for which any normal basis consists of r null and s non-null vectors, all mutually orthogonal. His result is as follows. Let L, M, N, L' denote the suffix-ranges $(1, 2, \cdots, r)$, $(r+1, \cdots, r+s)$, $(r+s+1, \cdots, n-r)$, $(n-r+1, \cdots, n)$, respectively. Then a canonical form for ds^2 is given by

$$[g_{ij}] = \begin{bmatrix} O & O & O & I \\ O & A & O & F \\ O & O & B & G \\ I & F' & G' & C \end{bmatrix},$$

where (i) the leading-diagonal submatrices are square (symmetric) of orders r, s, $n-2r-s$, r, (ii) I is unit $r \times r$, A, B are nonsingular, primes denote transposes, (iii) A, F, F' are independent of the coordinates x^α $(\alpha \epsilon L + N)$ and B, G, G' are independent of x^β $(\beta \epsilon L + M)$. A basis for the parallel $(r+s)$-plane is provided by the vectors $\delta_1{}^i, \delta_2{}^i, \cdots, \delta^i_{r+s}$ at

each point of V_n, the first r of which also provide a basis for its null part. As in the previous paper, the author explicitly restricts himself to the local problem in a V_n of class C^ω or C^u. *H. S. Ruse* (Leeds).

Sen, R. N. On an algebraic system generated by a single element and its application in Riemannian geometry. Bull. Calcutta Math. Soc. **42**, 1–13 (1950).

In earlier papers [cf. the same Bull. **41**, 113–120 (1949); these Rev. **11**, 399] the author has studied the parallel displacement of vectors in a Riemannian space containing an arbitrary affine connection. A certain system of affine connections, derived from the given connection, played a rôle in his work. In the present paper the author indicates how this system of affine connections may be considered as a particular embodiment of a certain abstract algebraic system whose properties are developed. *A. Fialkow.*

Rozenfel'd, B. A., and Abramov, A. A. Spaces with affine connection and symmetric spaces. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 72–147 (1950). (Russian)

A semi-expository account of the theory of affinely connected spaces and spaces of symmetry. It is "a systematic presentation of the theory of these spaces designed for mathematicians of other specialization, such an account being absent in the Russian as well as in foreign literatures." This rather extensive paper may be divided into three main parts. The first deals with various geometric spaces, viz.: the real linear vector space L_n and its conjugate space L_n^*; the equilinear space E_n, a linear vector space in which an exterior form for any n vectors is defined: $[x, y, \cdots, z] = e_{i_1 \cdots i_n} x^{i_1} \cdots z^{i_n}$, where e is a skew-symmetric tensor; B_n, a linear space with bilinear metric defining a scalar product $(x, y) = a_{ij} x^i y^j$; orthogonal space O_n which is a B_n with positive definite fundamental tensor a_{ij}, a pseudo-orthogonal space $^l O_n$ whose metric is $\sum_i {}^l \sigma_i (x^i)^2$ with $\overset{l}{\sigma_i} = -1$ for $i \leq l$ and $+1$ for $i > l$ (the group of automorphisms of this space is the pseudo-orthogonal group $^l O_n$ of the same index); the simplectic space Y_{2n} whose fundamental metric tensor is skew-symmetric and with simplectic group $\mathfrak{Y}_{2n}$, where $\sum_i (a_k{}^i a_i{}^{i+n} - a_k{}^{i+n} a_i{}^i) = \delta_{k, i-n} - \delta_{k-n, i}$. The most general space considered is the affine space with the group $\mathfrak{A}_n$, where $'x^i = a_j{}^i x^j + a^i$. The subgroup of $\mathfrak{A}_n$, $'x^i = x^i + a^i$, is the group of parallel displacements and the lines $x^i = a'^i t + b^i$ are the straight lines, t being the affine parameter determined up to a transformation $'t = at + b$. Three points on a line determine an invariant of $\mathfrak{A}_n$ called the ratio, i.e., $(t_1 - t_3)/(t_2 - t_3)$. If this ratio is 1 the points x_1, x_3 are symmetric with respect to x_2. Symmetries with respect to a fixed point define an involutory mapping given by $'x = 2a - x$, the product of two symmetries being a parallel displacement $('' x = x + 2(b-a))$. These concepts are then extended to differential geometric spaces by means of mappings of tangent spaces at nearby points. This leads to the development of tensor calculus and differential invariants for metric, affine, or equiaffine geometry. The second part is devoted to the theory of Lie groups and Lie algebras; their local and topological properties, isomorphisms, and representations are briefly, but clearly presented. The geometry is based primarily on the work of É. Cartan, the main theorem being that in any Lie group a unique affine connection without torsion is determined which is invariant under the group operations $x \to ax$; $x \to xa$; $x \to x^{-1}$. The particular groups presented in the first part are studied in detail and their affine structure is obtained from the corresponding group algebras. A space of

symmetry is now defined as an affine space without torsion, with each point of which is associated an involutory transformation, defined over the whole space, which leaves the affine connection invariant and generates a symmetry in the tangent space at the point. (For geodesic neighborhoods this is equivalent to the definition given in the first part.) The main theorem in this connection, also due to Cartan, is that a necessary and sufficient condition for a space to be one of symmetry is that $\nabla_e R_{jkl}{}^i = 0$. Other theorems discussed in some detail are that Lie groups regarded as affine spaces without torsion are spaces of symmetry and that a necessary and sufficient condition for a symmetry space to be a Lie group is that the fundamental group of the space contain a simply transitive subgroup. The last part is devoted to symmetry spaces whose fundamental group is simple. Their classification naturally depends on that of simple groups and these are discussed in considerable detail, except for the 5 isolated simple groups. The last section is devoted to realization or "models" of spaces of symmetry; this is really a brief resumé of the authors' work over the past half dozen years. *M. S. Knebelman.*

Ulanovskiĭ, M. A. On stationary groups of motions of spaces with linear projective and affine connection. Doklady Akad. Nauk SSSR (N.S.) **71**, 629–631 (1950). (Russian)

Let X_n be the n-dimensional space with linear projective connection given by the matrix $\|\omega_\beta{}^\alpha\|$ of linear differential forms $\omega_\beta{}^\alpha$ ($\alpha, \beta = 0, 1, \cdots, n, \omega_0{}^0 = 0$), where

$$\omega_\beta{}^\alpha(x, dx) = \Pi^\alpha_{\beta k}(x_1 \cdots x_n)dx^k, \qquad \omega_k{}^\alpha(x, x) = 0,$$
$$\omega_0{}^i(x, x) = \Pi^i_{0k}x^k = x^i, \qquad (i, k = 1, 2, \cdots, n).$$

The x^i are called the normal coordinates. Every motion of the space with linear projective connection, which has the point O $(0, \cdots, 0)$ of the normal coordinate system as an invariant point is given by the correspondence $x^{i\prime} = a_k{}^i x^k/(1 + a_k{}^0 x^k)$, $a_k{}^i = \text{constant}$, $a_k{}^0 = \text{constant}$. The matrix $\|A_\beta{}^\alpha\|$, where $A_0{}^0 = 1$, $A_0{}^i = 0$,

$$A_j{}^i = \frac{\partial}{\partial x^j}\left(\frac{a_k{}^i x^k}{1 + a_k{}^0 x^k}\right), \qquad A_i{}^0 = \frac{a_i{}^0}{1 + a_k{}^0 x^k}$$

transforms the matrix $\|\omega_\beta{}^\alpha(x, dx)\|$ in $\|\omega_\beta{}^\alpha(x', dx')\|$. The author proves the following theorem. When there exists in the space with linear projective connection a nontrivial motion, which does not change a point and also has all directions, there exists a neighborhood of this point where the given space is equivalent to the ordinary projective space. From this theorem we have as a consequence: The stationary group of motions in an n-dimensional space with linear projective connection depends on at most $n^2 - 1$ parameters. *F. Vyčichlo (Prague).*

Yano, Kentaro. Quelques remarques sur les groupes de transformations dans les espaces à connexion linéaire. I. Proc. Japan Acad. **22**, nos. 1–4, 41–47 (1946).
Yano, Kentaro, and Takano, Kazuo. Quelques remarques sur les groupes de transformations dans les espaces à connexion linéaire. II. Proc. Japan Acad. **22**, nos. 1–4, 69–74 (1946).

Let (1) $\bar{x}^\lambda = x^\lambda + \xi_b{}^\lambda \delta t$ be the infinitesimal transformations of an r-parametric group ($b = 1, \cdots, r$) and let $T \vdots\vdots (x)$ be any object whatsoever. The operator D_b induced by (1) is defined in the following way: $D_b T \vdots\vdots = T \vdots\vdots (\bar{x}) - \bar{T} \vdots\vdots (\bar{x})$. In particular, $D_b \xi^\lambda = (\xi_a{}^\alpha \xi^\lambda_{,\alpha} - \xi^\alpha \xi^\lambda_{,\alpha})\delta t$. Let ξ^b be a linear combination of the vectors $\xi_b{}^\lambda$ and (2) $\bar{x}^\lambda = x^\lambda + \xi^\lambda \delta t$ the

corresponding infinitesimal transformation. If $\Gamma^\nu_{\lambda\mu}$ are the coefficients of the connection of the space and $\bar{\Gamma}^\nu_{\lambda\mu}$ the transformed coefficients by (2), then (3) $\bar{\Gamma}^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} + D\Gamma^\nu_{\lambda\mu}$, where D is the operator belonging to ξ^λ. If v^ν is a vector field parallel with respect to $\Gamma^\nu_{\lambda\mu}$ and $\bar{v}^\nu = v^\nu + Dv^\nu$ is parallel with respect to $\bar{\Gamma}^\nu_{\lambda\mu}$ given by (3), then one says that the corresponding space admits a collineation whose infinitesimal transformation is (2). In the second part the authors find necessary and sufficient conditions for it in terms of different geometrical objects. Example: A necessary and sufficient condition that the space admit a collineation is that its infinitesimal transformation preserve the curvature tensor, torsion tensor, and their covariant derivatives.

V. Hlavatý (Bloomington, Ind.).

Yano, Kentaro. On the flat conformal differential geometry. III. Proc. Japan Acad. **22**, nos. 1–4, 9–19 (1946).
Yano, Kentaro. On the flat conformal differential geometry. IV. Proc. Japan Acad. **22**, nos. 1–4, 20–31 (1946).

The Frenet formulae in chapter II of this paper [same Proc. **21** (1945), 454–465 (1949); these Rev. **11**, 398] mentioned first in the review are not invariant under the homographic transformation of the corresponding projective parameter t (defined by a Schwartzian). Starting with the transformation formula (with respect to the homographic transformation of the projective parameter) for the first curvature one gets a conformal arc σ invariant under any homographic transformation of t which leads to purely conformal Frenet formulae. In the next section the same formulae are obtained by the Cartan method ["repère mobile" and the successive variation of secondary parameters] [for the Frenet formulae in a space which is not a conformally flat one, see the reviewer, Akad. Wetensch. Amsterdam, Proc. **38**, 281–286, 738–743, 1006–1011 (1935)]. In the fourth part a subspace $\xi^\nu = \xi^\nu(\eta^1, \cdots, \eta^m)$ of a flat conformal space is considered. Starting with the generator point $A_0{}^* = A_0$ of the subspace one defines the hyperspheres $A_i = (\partial \xi^\nu/\partial \eta^i)(\partial A_0/\partial \xi^\nu)$, $i = 1, \cdots, m$, the $n - m$ mutually orthogonal hyperspheres A_P ($P = m+1, \cdots, n$) going through A_0 and tangent to the subspace and the point hypersphere $A_{\bar\omega}$ which is the intersection point ($\neq A_0$) of A_i and A_P. The derivatives of the spheres A_0, A_i, A_P, $A_{\bar\omega}$ may be expressed as linear combinations of A_0, A_i, A_P, $A_{\bar\omega}$, depending on the metric tensor g_{ij} of the subspace, and on its second and third tensors M_{jkP}, L_{PQk}, respectively. The integrability conditions of these fundamental equations show that tensors g_{ij}, M_{jkP}, L_{PQk} satisfying them define a subspace up to conformal transformations. [For the theory of a subspace in a conformal space which is not a conformally flat one, cf. the above mentioned papers by the reviewer.]

V. Hlavatý (Bloomington, Ind.).

Ide, S. On the theory of curves in an n-dimensional space with the metrics $S = \int\{A_i(x, x')x^{\prime i} + B(x, x')\}^{1/n}dt$. Tensor **9**, 25–29 (1949). (Japanese)

Following the reviewer's theory [Trans. Amer. Math. Soc. **44**, 153–167 (1938)], the curve theory in the space of the title is discussed. The main purpose is to find Frenet's formulas and the invariants of a curve. There are two kinds of Frenet formulas and of invariants, that is, the ones for the absolute derivatives of the tangent vector $x^{\prime i}$ and the others for those of the covariant vector A_i. At the end, the relations between the two kinds of invariants are obtained.

A. Kawaguchi (Sapporo).

Tonowoka, K. On invariants of

$$\int_{(n-1)} (A_i{}^{\alpha(2)}{}^{\beta(3)}p^i{}_{\alpha(2)}p^j{}_{\beta(3)} + B_i{}^{\beta(3)}p^j{}_{\beta(3)} + C)^{1/\nu}du^1du^2 \cdots du^{n-1}.$$

Tensor 9, 18–24 (1949). (Japanese)

Under the assumption that the $(n-1)$-ple integral in the title, where $p^i{}_{\alpha(2)}=\partial^2x^i/(\partial u^{\alpha_1}\partial u^{\alpha_2})$, $p^i{}_{\beta(3)}=\partial^3x^i/(\partial u^{\beta_1}\partial u^{\beta_2}\partial u^{\beta_3})$ and A, B, C are all functions of x^i and $p_\alpha{}^i=\partial x^i/\partial u^\alpha$, the author finds a relative scalar function $L(x^i, p_\alpha{}^i)$ derived from only the functions A in the integrand. Then the metric tensor, connection parameters, and absolute derivatives are obtained from the function L in a similar (but slightly different) method to that of É. Cartan [Les espaces métriques fondés sur la notion d'aire, Actualités Sci. Ind., no. 72, Hermann, Paris, 1933]. *A. Kawaguchi* (Sapporo).

Su, Buchin. A generalization of descriptive collineations in a space of K-spreads. J. London Math. Soc. 25, 236–238 (1950).

The author extends the theory of descriptive collineations in a space of K-spreads as done by R. S. Clark [Proc. Cambridge Philos. Soc. 41, 210–223 (1945); these Rev. 7, 175] to the case in which the functions $\xi^i(x, p)$ defining the transformation depend upon the orientation as well as on position. There is a generalization of the operation of Lie derivation already used by the author in a previous paper [Acad. Sinica Science Record 2, 139–146 (1948); these Rev. 10, 149]. *E. T. Davies* (Southampton).

Raševskiĭ, P. K. Galois theory in fields of geometric objects. Trudy Sem. Vektor. Tenzor. Analizu 7, 167–186 (1949). (Russian)

The author considers geometric objects and (possibly infinite) Lie groups of transformations in a space X_n homeomorphic with a region in n-space R^n; X_n carries with it a set of coordinate systems (termed admissible) such that the passage from one coordinate system to another is effected by analytic functions. A geometric object φ is termed "rational" if after an admissible change in coordinate system the new components of φ are rational functions of the old components and of the partial derivatives of various orders of the new coordinates with respect to the old. Each Lie group is assumed to be given by a system of differential equations in unknowns $\bar{x}_1, \cdots, \bar{x}_n$ of the form $F=0$, where F is a polynomial in the partial derivatives $\partial^{i_1+\cdots+i_n}\bar{x}_j/\partial^{i_1}x_1 \cdots \partial^{i_n}x_n$ $(1 \leqq j \leqq n, i_1+\cdots+i_n \geqq 1)$ with coefficients analytic in $x_1, \cdots, x_n$, and $\bar{x}_1, \cdots, \bar{x}_n$ regular throughout X_n [here (x_i) is a fixed coordinate system]; a Lie group of this sort for which the system $F=0$ satisfies certain other conditions is termed "rational." It is shown that for any rational Lie group G there exists a rational geometric object φ which is invariant under (every transformation in) G and under no analytic transformation not in G. Given two coordinate systems (x_i), (t_i) for X_n, the components (with respect to (x_i)) of a rational geometric object ψ are analytic functions of (t_i); the author shows that if ψ is invariant under G then the components of ψ are rational functions of the components of φ and their partial derivatives with respect to t_i of various orders. The set of all geometric objects which can be expressed in this way in terms of a single rational geometric object φ is called a "field of geometric objects." If this φ admits a transitive Lie group the field is called "transitive." It then is proved that there is a one-to-one correspondence between transitive fields of geometric objects and rational Lie groups. The proofs make use of results which to the reviewer's knowledge have not been completely established. *E. R. Kolchin.*

Varga, O. Über den Zusammenhang der Krümmungsaffinoren in zwei eineindeutig aufeinander abgebildeten Finslerschen Räumen. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 132–135 (1950).

The author derives formulas for the difference of the curvature tensors of two metrical connections given on the same space of line elements, and related formulas.

S. Chern (Chicago, Ill.).

NUMERICAL AND GRAPHICAL METHODS

Blanch, Gertrude, and Siegel, Roselyn. Table of modified Bernoulli polynomials. J. Research Nat. Bur. Standards 44, 103–107 (1950).

Values of $b_k(x) = \frac{1}{2}[(-2\pi)^k/k!]B_k(x/2\pi)$ are given to 17 decimals for $k=1(1)11$, $36x/\pi=0(1)36$. The polynomials for $k=1(1)11$ are given explicitly in powers of x. *J. C. P. Miller* (London).

Anscombe, F. J. Table of the hyperbolic transformation $\sinh^{-1}\sqrt{x}$. J. Roy. Statist. Soc. Ser. A. 113, 228–229 (1950).

A table of the hyperbolic transformation $\sinh^{-1}x$ is constructed for $x=0.00(0.01)1.0(0.1)10(1)200(10)590$, where $x=a(b)c$ means that x goes from a to c by steps of size b. The function of this table is to transform counts from skew distributions in order to stabilize the variance and permit the analysis of variance. It may also be used to normalize Student's t by the transformation $y=\pm\sinh^{-1}(3t^2/2n)^{\frac{1}{2}}$. *H. Chernoff* (Urbana, Ill.).

MacDonald, A. D. Tables of the function $e^{-\alpha z/\gamma}M(\alpha; \gamma; z)$. Canadian J. Research. Sect. A. 28, 175–179 (1950).

This paper gives six-figure tables of $\mu(z)=e^{-\alpha z/\gamma}M(\alpha, \gamma, z)$, where $M(\alpha,\gamma,z)=1+[\alpha/\gamma]z+[\alpha(\alpha+1)/\gamma(\gamma+1)]z^2/2!+\cdots$ is the confluent hypergeometric function. Values are given for $\alpha=0.001, 0.01, 0.1(0.1)1.0, 0.25, 0.75$ and $\gamma=0.5(0.5)2.0$, $z=0(0.1)1(0.5)8$. The tables supplement those in an earlier paper by the author [J. Math. Physics 28, 183–191 (1949); these Rev. 11, 246], where references to earlier tables are given. *J. C. P. Miller* (London).

✶Table of Characteristic Values of Mathieu's Differential Equation. AMP Report 165. 1R, Prepared by the Mathematical Tables Project, National Bureau of Standards, 1945. xxiv+39 pp.

Mathieu's equation, which is taken in the form

$$d^2y/dt^2 + (b - s\cos^2 t)y = 0,$$

has solutions of period π or 2π only for a discrete, though denumerably infinite, set of values of b for each particular value of s; these values of b are the characteristic values and are denoted in the present table by be_r and bo_r for even and odd solutions y, respectively, the suffix r corresponding to exactly $2r$ zeros per period 2π in t. The tables give values of be_r for $r=0(1)15$ and bo_r for $r=1(1)15$ to 8 decimals and for s ranging from 0 to 100 by various intervals 0.2, 0.5 when r and s are small, up to 4 when both are large. In all cases modified second differences are given which are ade-

quate for interpolation to 8-figure accuracy. A two-page table is also provided, giving the second difference coefficients $E_2(p)$ and $F_2(p) = E_2(1-p)$ for use in Everett's interpolation formulae. An introduction gives some properties of the functions and describes methods of computation and interpolation; numerical illustrations are given.

Mathieu's equation is more usually taken in the more symmetrical form $d^2y/dt^2 + (a - 2\theta \cos 2t)y = 0$ [see, for example, Ince, Proc. Roy. Soc. Edinburgh. Sect. A. **52**, 355–423 (1932)], or $d^2y/dt^2 + (a - 2q \cos 2t)y = 0$ as advocated by McLachlan [Theory and Application of Mathieu Functions, Oxford, 1947, pp. vi, 1; these Rev. **9**, 31]. The relations $a = \alpha = b - \frac{1}{2}s$, $y = \theta = \frac{1}{2}s$ allow characteristic values, usually denoted by $a_r = \alpha_r$, and $b_r = \beta_r$, for the latter forms to be read from these tables.   *J. C. P. Miller* (London).

✳High-Speed Computing Devices. By the Staff of Engineering Research Associates, Inc. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1950. xiii+451 pp. $6.50.

This volume is primarily a discussion of the mechanical devices and electrical circuits which can be incorporated into computing machines. It also includes a brief survey of numerical analysis and a detailed discussion of two problems. One of these is examined from the points of view of determining desirable commands and illustrating the processes of programming and coding. The other is concerned with the possibility of using digital computers to determine the position of a target tracked by hyperbolic instruments. Descriptions of some existing computing systems are given and there are extensive bibliographies.   *J. Todd.*

Raymond, François-Henri. Sur un type général de machines mathématiques algébriques. Ann. Télécommun. **5**, 2–20 (1950).

The theory of three continuous type mathematical machines is described without a technical description of the machines themselves which will appear in the future. One of these is essentially a linear equation solver of Goldberg-Brown type [cf. J. Appl. Phys. **19**, 339–345 (1948); these Rev. **9**, 535]. Their stability theory is given as well as a list of applications. The Redheffer formula [Quart. Appl. Math. **6**, 342–343 (1948); these Rev. **10**, 152] for the error in a system of linear equations is derived and, based on this, an error formula for the given machine is set up. However, the reviewer is unable to see how the argument takes into account the scaling operations necessary for using the machine or the justification for the assumption that the error in the coefficients is proportional to the coefficients. The coefficients are entered on helical potentiometers and it is customary to assume that the error is a fraction of the full scale. The other two machines are differential analyzers for systems of differential equations with constant coefficients. Thus if x is an n-dimensional vector, A, B, C, and D matrices with real coefficients, and p denotes differentiation, then the machine will solve the equation $(Ap^3 + Bp^2 + Cp + D)x = a$, or rather an integrated version of this. Integration is by means of a feedback amplifier. The stability theory for this device is given and under certain simplifying assumptions the effect of the amplifier on the characteristic frequencies of the solution is described. However, noise is not considered. An addition to the last two machines in the form of a servo-motor multiplier is described and its use will also be given in a future paper   *F. J. Murray* (New York, N. Y.).

Turing, A. M. Computing machinery and intelligence. Mind **59**, 433–460 (1950).

Berkeley, Edmund C. The relations between symbolic logic and large-scale calculating machines. Science (N.S.) **112**, 395–399 (1950).

Sadosky, Manuel. Recent progress and evolution of mechanical and automatic calculation. Ciencia y Tecnica **115**, 170–186 (1950). (Spanish. French summary)

Shaw, Robert F. Arithmetic operations in a binary computer. Rev. Sci. Instruments **21**, 687–693 (1950).

This article contains elementary considerations of the operations of binary arithmetic, from the point of view of mechanization, with special reference to the Eckert-Mauchly BINAC (a general-purpose binary computer).   *J. Todd* (Washington, D. C.).

Forsythe, George E. Note on rounding-off errors. National Bureau of Standards, Los Angeles, Calif. 3 pp. (1950).

In the theoretical discussion of the accumulation of rounding-off errors in integration processes it is customary to assume that they behave as independent, uniformly distributed random variables [cf., e.g., Rademacher, Proceedings of a Symposium on Large-Scale Digital Calculating Machinery, Annals of the Computation Laboratory of Harvard University, v. 16, pp. 176–187, 1948, these Rev. **9**, 468]. This assumption does not appear to be valid in practice [cf. Huskey, J. Research Nat. Bur. Standards **42**, 57–62 (1949); these Rev. **11**, 266]. In order that the theory can be saved the author proposes the following process for the rounding (to an integer) of a real number u: round up to $[u]+1$ with probability $v = u - [u]$, round down to $[u]$ with probability $1-v$, the choice being made by some independent chance mechanism. The error ϵ is then a random variable with $E(\epsilon) = 0$ and $E(\epsilon^2) = v(1-v) \leq \frac{1}{4}$. The last bound is to be compared with the usual assumption $E(\epsilon^2) = 1/12$.   *J. Todd* (Washington, D. C.).

✳Happach, V. Ausgleichsrechnung. 2d ed. B. G. Teubner, Leipzig, 1950. 104 pp. $1.50.

This is a "practical" treatment of the method of least squares. No proofs are given. The author attempts to explain the techniques by working out many numerical examples drawn from science, industry, and engineering. The author makes no use or mention of the important advances in estimation theory that have been taking place for more than 20 years.   *B. Epstein* (Detroit, Mich.).

Wolf, Helmut. Triangulation adjustment. General discussion and new procedure. Bull. Géodésique N.S. **1950**, no. 16, 87–104 (1950).

Levallois, J. J., et Dupuy, M. Sur le calcul des grandes géodésiques. Bull. Géodésique N.S. **1950**, no. 16, 105–117 (1950).

Exposé sommaire des méthodes de calcul, élaborées à l'Institut Géographique National.
                                        Extract from the paper.

Basile, R., et Janin, R. Résolution de systèmes d'équations linéaires algébriques et inversions de matrices au moyen des machines de mécanographie comptable. O.N.E.R.A. Publ. no. 28, vi+21 pp. (4 plates) (1949).

Gavurin, M. K. The use of polynomials of best approximation for improving the convergence of iterative processes. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 156–160 (1950). (Russian)

Let the nth order matrix A have linear elementary divisors and real latent roots such that

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \geqq |\lambda_4| \geqq \cdots \geqq |\lambda_n|.$$

A method is proposed whereby, if λ_1 and λ_2 have been found approximately, then the estimate for λ_1 can be improved. To begin with, expressions of the form $f_k = \sum_{i=1}^k \alpha_i \lambda_i^k$ are derived, e.g., from the iterates of A. First approximations to λ_1 and λ_2 are then f_{p+1}/f_p, $(f_{k+1} - \lambda_1 f_k)/(f_k - \lambda_1 f_{k-1})$. The second approximation to λ_1 is then θ_{p+1}/θ_p, where $\theta_p = \sum_{k=0}^p c_k f_k$; here the c_k are chosen so as to maximise the predominance in θ_p of the terms in λ_1, and are essentially the coefficients of the Čebyšev polynomial $T_p(x)$ transformed to the interval $(-|\lambda_2|, |\lambda_2|)$. The error of the resulting estimate is compared with that of Aitken's [Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)].

The second part is devoted to the approximate solution of $X - AX = Y$, assuming that $|\lambda_i| < 1$, $i = 1, 2, \cdots, n$, so that $X = \sum_{k=0}^\infty A^k Y$. The proposed approximation is $Z_p = \sum_{k=0}^p c_k A^k Y$, where the c_k are now such that

$$S_p(\lambda) = \sum_{k=0}^p c_k \lambda^p$$

is the polynomial of degree p of best approximation to $(1-\lambda)^{-1}$ over the interval $(-|\lambda_1|, |\lambda_1|)$, and is expressible in terms of Čebyšev polynomials. More generally, one may approximate to $(1-\lambda)^{-1}$ over any set of intervals known to include all the λ_i, even if they do not all lie in $(-1, 1)$. It is claimed that the method also applies in Hilbert spaces.

F. V. Atkinson (Ibadan).

Fettis, Henry E. A method for obtaining the characteristic equation of a matrix and computing the associated modal columns. Quart. Appl. Math. **8**, 206–212 (1950).

A method is presented for the calculation of the coefficients $b_1, \cdots, b_n$ of the characteristic equation

$$\lambda^n - b_1 \lambda^{n-1} + \cdots + (-1)^n b_n = 0$$

of a matrix A, based on the recursion formula

$$b_{k+1} = (\operatorname{Tr} A_k)/k + 1,$$

where $A_k = b_k A - AA_{k-1}$, $k = 0, 1, \cdots, n-1$. By the use of the above procedure it is also possible to calculate, with relative ease, the characteristic vectors of the given matrix, since a given characteristic vector $x^{(i)}$ is proportional to any column of the matrix

$$A\lambda_i^n - A_1\lambda_i^{n-1} + A_2\lambda_i^{n-2} - \cdots + (-1)^{n-1}\lambda_{n-1}\lambda_i,$$

where λ_i is the corresponding characteristic root. The procedure is illustrated by a numerical example.

H. Polachek (White Oak, Md.).

Teodorčik, K. F. An iterative method of solution of the characteristic equation of systems of the third order. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 231–234 (1949). (Russian)

The equation $x^3 - x^2 + ax - b = 0$, $a > b > 0$, can be solved by an iterative procedure. Let $x_1 = b/a$, define x_{n+1} by $x_{n+1} = F(x_n)$, where $F(x) = (x^3 + b)/(x^2 + a)$. Then the sequence

x_n converges to a root of the equation. The method is applied to linear differential equations of order 3 with constant coefficients.

J. G. Wendel (New Haven, Conn.).

Bonneau, Eugène. Une méthode nouvelle pour le calcul des racines complexes des équations algébriques à coefficients réels. C. R. Acad. Sci. Paris **231**, 99–101 (1950).

Let the real polynomial $P_0(z) = \sum_0^n A_{j,0} z^{n-j}$, $A_{n,0} \neq 0$, have the zeros z_j with $|z_n| \leqq |z_{n-1}| \leqq \cdots \leqq |z_1|$ and with the z_j that lie in the closed upper half plane simple zeros possessing distinct moduli. According to the Graeffe method, form the polynomials $P_k(z) = \sum_0^n A_{j,k} z^j$ with zeros $-z_j^K$, $K = 2^k$. It is known that, if z_j is real, then $(A_{j,k}/A_{j-1,k})^{2/K} \to |z_j|^2$ as $k \to \infty$, whereas, if z_j is nonreal and is the conjugate imaginary of z_{j+1}, then $(A_{j+1,k}/A_{j-1,k})^{1/K} \to |z_j|^2 = |z_{j+1}|^2$. For the latter case, the present note gives the following formula for the real part of z_j: $|z_j|^2 \lim_{k\to\infty} 2^{-k-1}(C_{j+1,k} - C_{j-1,k})$. The constants $C_{j,k}$ are obtained by considering the sequence $Q_k(z)$ of Graeffe polynomials corresponding to $Q_0(z) = P_0(z-t)$ for small values of t and, essentially, by using the approximations $Q_k(z) \sim \sum_0^n A_{j,k}(1 + tC_{j,k}) z^j$ obtained by discarding terms in the higher powers of t. The proof is based upon the relation $z_j + \bar{z}_j = [(d/dt)|z_j + t|^2]_{t=0}$. *M. Marden.*

Meulenbeld, B. Note on the representation of the values of polynomials with real coefficients for complex values of the variable. Nederl. Akad. Wetensch., Proc. **53**, 956–958 = Indagationes Math. **12**, 351–353 (1950).

The author gives a graphical method for finding the value of a real polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $a_0 \neq 0$, for a given complex z. First, a directed polygonal line $P_0 P_1 \cdots P_{n+1}$ is constructed with the segment $P_k P_{k+1}$ of length $|a_k|$ and each angle $P_{k-1} P_k P_{k+1}$ of magnitude φ, independent of k. An arbitrary point A_1 is then selected and the vector $A_1 P_1$ is taken as $a_0 z$. Now the triangles $A_1 P_2 A_2$, $A_2 P_3 A_3$, $\cdots$, $A_{n-1} P_n A_n$ are constructed all similar to the triangle $P_0 P_1 A_1$. Finally, the vector $A_n P_{n+1}$, taken relative to $P_n P_{n+1}$ as real axis, represents $f(z)$.

M. Marden (Milwaukee, Wis.).

Schäfer, Otto, und Lander, Gerhard. Ein elektrisches Gerät zur Berechnung von Produkt-Integralen. Arch. Elektr. Übertragung **4**, 59–64 (1950).

An instrument is described for calculating integrals of the form $\int_a^b f'(t) F(x-t) dt$, where $f(t)$ (or $f'(t)$) and $F(t)$ are empirically given functions and x is a constant parameter. The multiplication and integration are performed by a device similar to an induction motor, into which are fed two alternating currents whose amplitudes are proportional to $f'(t)$ and $F(x-t)$, respectively. These currents are the amplified outputs of two photocells, which read by optical means the graphs (prepared as opaque paper cut-outs) of the given functions. If the ordinate of the function is to be read, a vertical slit in front of the required ordinate of the cut-out allows an amount of light to fall on the photocell behind the cut-out which is proportional to the ordinate. This serves to modulate alternations in the light intensity which are produced by an oscillating mirror between the light source and the slit. If the derivative of the function is desired, the slit is omitted and the source (a vertical line filament) is focused on the ordinate of the graph at the desired value of t. The (small) oscillations of the mirror produce corresponding oscillations of the filament image to each side of this value of t and hence the light reaching the

photocell has fluctuations proportional to the slope of the curve. Applications of the instrument are also discussed.
P. W. Ketchum (Urbana, Ill.).

Tibiletti, Cesarina. Procedimenti grafici per l'integrazione delle equazioni differenziali. Period. Mat. (4) **28**, 98–113 (1950).

This is mainly an expository paper describing some of the classical methods of integrating differential equations graphically. For first-order equations the methods of isoclinal curves and of envelope curves are explained. For second-order equations, methods based on the point-by-point construction of centers and radii of curvature are treated. A few other types are mentioned, especially the so-called method of averaging. *W. E. Milne.*

Hartree, D. R. The calculation of atomic structures. Reports on Progress in Physics **11**, 113–143 (1948).

A report on approximate methods of solving the wave equation for problems of atomic structure.

✶**Serebrennikov, M. G.** Garmoničeskiĭ analiz. [Harmonic Analysis]. OGIZ. Moscow-Leningrad, 1948. 504 pp.

(I) Theoretical foundations of harmonic analysis; (II) Practical harmonic analysis; (III) Examples of the application of harmonic analysis in technology and science; Two appendices with tables. *Table of contents.*

Boulanger, Georges. Sur quelques propriétés de structure des abaques à plans superposés. Bull. Soc. Math. Belgique **2** (1948–1949), 41–48 (1950).
Expository paper.

ASTRONOMY

de Jekhowsky, Benjamin. Démonstration simple et directe du théorème d'Euler relatif aux orbites paraboliques. C. R. Acad. Sci. Paris **230**, 1738–1740 (1950).

Rubbert, Friedrich Karl. Direkte Integration des Zweikörperproblems. Astr. Nachr. **276**, 238–240 (1948).
Rubbert, F. K. Direkte Integration des Zweikörperproblems. II. Astr. Nachr. **277**, 112–114 (1949).

In these papers the author derives from the differential equations of the two-body problem several terms of the series expansions in powers of the eccentricity e, $e < 1$, for the functions r and $\cos \vartheta$, where r and ϑ are polar coordinates of one body with respect to the center of gravity of the system. These series have been derived previously, by somewhat similar methods, by F. R. Moulton [An Introduction to Celestial Mechanics, Macmillan, New York, 1914, chapter V] and by W. D. MacMillan [Theoretical Mechanics, Statics and the Dynamics of a Particle, McGraw-Hill, New York, 1927, chapter XII]. The author briefly mentions generalizations of the law of attraction; fuller discussions of such generalizations have been given by various writers, for example, by MacMillan [loc. cit.] and by E. J. Moulton and F. H. Hodge [Amer. Math. Monthly **15**, 119–130 (1908)]. *E. J. Moulton* (Evanston, Ill.).

Rubbert, F. K. Zur Integration des Zweikörperproblems. Astr. Nachr. **278**, 105–114 (1950).

The problem is first reduced to a differential equation of the first order for the reciprocal distance. The integration is performed with the aid of a method analogous to the introduction of the elliptic functions sn x and cn x. Kepler's equation for elliptic motion, $\tau = u - e \sin u$, defines the inverse function, $u = \text{kep} (\tau, e) = \text{kep} \tau$. The coordinates in elliptic motion as function of the time are then expressed in closed form with the aid of sk $\tau = \sin \text{kep} \tau$ and ck $\tau = \cos \text{kep} \tau$. The properties of these functions are examined and developments in series are derived. The coordinates in hyperbolic and parabolic motion are similarly treated.
D. Brouwer (New Haven, Conn.).

Lattermann, Karl. Über die Nichtexistenz periodischer Lösungen in der Nähe der kritischen Kreise. Math. Ann. **121**, 327–339 (1950).

This paper is an application of the method of integrodifferential equations for periodic solutions of differential equations. This method was introduced in celestial mechanics by Hölder [Math. Z. **31**, 197–257 (1929)]. By solving the system of integro-differential equations by successive approximations, he showed that in the restricted three-body problem where $n_1/n_2 = (q+1)/q$ there are no periodic solutions of the first kind possible if q is an integer. The proof was given by showing that at least one of the "Verzweigungsgleichungen" (which express Hamilton's principle of least action) is unequal to zero. In this paper the author extends this to the plane three-body problem. He mentions that in the case of small inclinations in the perturbed motion nothing will be changed as to the fundamental significance of his result. [Reviewer's note. The entire method is very unfamiliar in the field of practical celestial mechanics. It appears, however, that it may prove useful in dealing with stability problems in the theories of minor planets and shortperiod comets.] *A. J. J. van Woerkom.*

Hil'mi, G. F. The problem of n bodies in celestial mechanics and cosmogony. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **14**, 46–50 (1950). (Russian)

The paper is concerned with two groups of problems in mechanics of gravitating bodies: (i) investigation of necessary and sufficient conditions under which the system of n bodies is stable, semi-stable, or completely dissipative for $t \to +\infty$ as well as for $t \to -\infty$; (ii) study of the possibility of the combination of a completely dissipative regime in one direction of time with a semi-stable regime for the opposite direction of the time. Some new theorems are announced and their cosmogonical significance is briefly discussed. No proofs are given. [Cf. Šmidt, Doklady Akad. Nauk SSSR (N.S.) **58**, 213–216 (1947); Hil'mi, ibid. (N.S.) **62**, 39–42 (1948); Šmidt, ibid. (N.S.) **62**, 43–46 (1948); Šmidt and Hil'mi, Uspehi Matem. Nauk (N.S.) **3**, no. 4(26), 157–159 (1948); these Rev. **10**, 487]. *E. Leimanis.*

Hagihara, Yusuke. On the general theory of libration. Jap. J. Astr. Geophysics **21**, nos. 1–2, 29–43 (1945).

The author develops a generalized form of Poincaré's treatment of the theory of libration [Leçons de Mécanique Céleste, v. 1, Gauthier-Villars, Paris, 1905, p. 364]. This more general theory is applicable to the case of nearly commensurable mean motions and employs an extension of Hirayama's criterion for libration [Astr. J. **38**, 147–148 (1928)]. *R. G. Langebartel* (Saltsjöbaden).

Hagihara, Yusuke. A proof of Poisson's theorem on the invariability of the major-axes of planetary orbits. Jap. J. Astr. Geophysics 21, nos. 1–2, 9–27 (1945).

By working with the relative canonical coordinates of Jacobi and using an induction argument on the order of approximation the author proves the theorem of Poisson simultaneously with the theorems of Poincaré on ranks and classes [Leçons de Mécanique Céleste, v. 1, Gauthier-Villars, Paris, 1905, pp. 131, 343]. *R. G. Langebartel.*

Matukuma, Takehiko. Motion of perihelion and node of the planets. Sci. Rep. Tôhoku Univ., Ser. 1. 33, 115–121 (1949).

The author computes without giving details the mean perihelion motion of the Earth and Venus and the mean node motion of the Earth, Venus, and Mars according to the formulation of Weyl [Amer. J. Math. 60, 889–896 (1938)] using Stockwell's constants [Memoir on the secular variation of the elements of the orbits of the eight principal planets, Smithsonian Contributions to Knowledge, v. 18, 1873]. *R. G. Langebartel* (Saltsjöbaden).

Shen-Zee. A sequel to "A method of computing general perturbations of the asteroids." Jap. J. Astr. Geophysics 19, 1–8 (1941).

The equations in the author's previous paper [same J. 18, 1–44 (1940)] derived for the perturbations of an asteroid due to an exterior planet are modified for the case of the asteroid exterior to the planet. *R. G. Langebartel.*

Shen-Zee. A literal expansion of the disturbing function practically applicable to computing general perturbations of the Trojan group of asteroids due to Jupiter. Jap. J. Astr. Geophysics 19, 217–224 (1942).

The perturbational function is expanded in a form adapted for treatment of the case when the angle subtended at the sun by the two planets is always larger than about 40°. It could be applied, for example, to the perturbation of an asteroid of the Trojan group by Jupiter, or of 944 Hidalgo by Jupiter, or of 433 Eros by Mars, or of the mutual perturbation of Pluto and Neptune. *R. G. Langebartel.*

Herrick, Samuel. A modification of the "variation-of-constants" method for special perturbations. Publ. Astr. Soc. Pacific 60, 321–323 (1948).

Zagar, Francesco. Sui movimenti interni negli ammassi stellari sferici. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 5 (1947/48), 121–139 (1949).

It is assumed that the stellar density ρ in a spherical cluster varies as $[a^2/(a^2+r^2)]^{5/2}\rho_0$, where a is a constant, r the distance from the center of the cluster, and ρ_0 the density at the center of the cluster ($r=0$). The mass contained in a sphere of radius r is easily obtained by integrating the expression for $\rho(r)$, and Newton's law yields the force which acts upon a star of mass m at a distance r from the center. We thus get the equation of motion, from which the area and the energy integrals are derived. Putting $dr/dt=0$ in the energy integral, we obtain an algebraic equation in r whose roots are the distances at which the velocity of the star under consideration is either a maximum or a minimum. According to the sign of the arbitrary constant h in the energy integral (i.e., according to the initial velocity of the star), the equation has either one or two real, positive roots. For positive values of h there is only a minimum distance,

after which the star will recede indefinitely and leave the cluster forever; for negative values of h the motion will be oscillatory in r (coronal motion.) In the latter case the star may move entirely within the confines of the cluster or may move alternately in and out of it; restrictive conditions are found for the motion of the star to be limited within the radius of the cluster. A table gives the escape velocities of cluster stars as function of r, assuming observational values for the various constants. The velocities thus found are relatively small (from 1 to 4 km/sec); in other words a globular cluster has a chance to survive only if the velocity of its member stars is exceedingly small. A change in the exponent of the ρ distribution formula from 5/2 to 1/2 does not change things materially.

In the second part of his paper, the author examines the form of the star trajectories inside a globular cluster in the absence of perturbing forces. Eliminating the time from the area and energy integrals, the equation of the trajectory is obtained as

$$\pm(\theta-\theta_0) = \int (A\cos\varphi + B - \cot^2\varphi)^{-\frac{1}{2}}\sin^{-2}\varphi\,d\varphi,$$

where θ is the angle at the center, $\tan\varphi = r/a$, A and B are constants. It is shown that for large values of r the orbit approaches a conic section. For small values of r the author, spurning any numerical-integration approach, goes to considerable trouble trying to expand the integral in series and successfully obtains a very complicated expression which, with suitable initial conditions, represents something like a precessional ellipse. In view of the small probability that a star will move free of perturbation inside a globular cluster, this last section of the paper seems to have a purely academic value. *L. Jacchia* (Cambridge, Mass.).

Gião, Antonio. Sur le mouvement général de la matière à échelle cosmologique. C. R. Acad. Sci. Paris 231, 605–606 (1950).

Zappa, L. I problemi di idrodinamica relativi a masse di dimensioni cosmiche. Mem. Soc. Astr. Ital. (N.S.) 21, 149–157 (1950).

Prasad, C. On the stability of Maclaurin spheroids rotating with constant angular velocity. Quart. J. Math., Oxford Ser. (2) 1, 117–121 (1950).

The author attempts to give a more precise form to a statement of Jeans [Astronomy and Cosmography, 2d ed., Cambridge University Press, 1929, p. 194] concerning the change of stability of Maclaurin ellipsoids. Other discussions [see, for example, P. Appell, Traité de mécanique rationelle, v. 4₁, Gauthier-Villars, Paris, 1921] are not mentioned. *W. S. Jardetzky* (New York, N. Y.).

Ekimov, V. V. The exact expressions of the normal gradient of gravity and of its components. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. 4, no. 3(56), 103–125 (1949). (Russian)

This paper discusses the approximate expression of the potential of a rotating ellipsoid. The author gives a detailed deduction of two formulas for the second derivatives $\partial^2 U/\partial x\partial z$ and $\partial^2 U/\partial z^2$ ($\partial^2 U/\partial y\partial z = 0$ if the ellipsoid is one of revolution), the first of which is new and the second identical with the known Bruns' formula as the author himself shows. *E. Kogbetliantz* (New York, N. Y.).

Feĭgel'son, E. M. The distribution in height of the temperature of the Earth's atmosphere taking account of radiative and vertical turbulent heat exchange. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 14, 359–382 (1950). (Russian)

Plutôt que d'utiliser les équations approchées de Schwarzschild, l'auteur établit les équations exactes d'échange radiatif de chaleur en présence d'une conductibilité thermique turbulente verticale. En se donnant des conditions aux limites à la surface de la terre et des conditions à l'infini, l'auteur résoud le système par des approximations successives, par une méthode analogue à celle de Chandrasekhar. L'auteur calcule les quatre premières approximations et montre que les températures en altitude obtenues en quatrième approximation dépassent de 6 à 10° celles qu'on obtient par la méthode de Schwarzschild. Les résultats numériques sont en assez bon accord avec les observations de Boutaric et Robitsch. *M. Kiveliovitch* (Paris).

MECHANICS

Delone, B. N. On a duplicator linkage of Prof. N. B. Delone. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 2, 101 (1947). (Russian)

Let A, B, C, D, E be hinges, $AB = BC = CD = DE$, A and E fixed. If the positions C_1, C_2 are symmetric relative to the center O of AE, the corresponding positions M_1 and M_2 coincide. If C describes a circle about O, M will describe the same closed curve twice. The inventor's son gives a four-line proof of this fact. *A. W. Wundheiler.*

Semenov, M. V. The connecting-rod curves of four-bar linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 10, 31–70 (1947). (Russian)

This paper introduces a general method of construction of connecting-rod curves (x, y) based on the Fourier representation

$$x_k + iy_k = \sum_k [R_k' \exp ik(\psi_k' + \phi) + R_k'' \exp ik(\psi_k'' - \phi)],$$

where ϕ is the azimuth of the driving-crank. The method consists in the determination of the "revolving vectors" (R', ψ') and (R'', ψ'') for each "harmonic," for several ϕ values, and adding them to obtain the $x_k + iy_k$ vector. The method is applied to seven four-bar mechanisms with turning (T) or sliding (S) pairs. The pair sequences studied are $TTSS$ and $TTTS$ (S sliding towards the adjacent T) with all (nontrivial) choices of the fixed link. On all but one of these mechanisms the expansion breaks off for $k > 2$ or $k > 3$; in one mechanism ($STTT$, based on ST: the familiar slider-crank of the piston engine) the usual approximation is applied and k reduces then to 0, 1, 2. A detailed description of the connecting-rod curves is given for each case, and diagrams (based on 12 crank positions per revolution) are given. An error analysis is given for the piston linkage. Special cases are discussed and technological applications (to knitting machinery, quick return and intermittent motion) are given. There is also some discussion of the approximation of prescribed curves. *A. W. Wundheiler* (Chicago, Ill.).

Bach, K. Die Verwirklichung vorgegebener Winkelgeschwindigkeitsgesetze bei Doppelkurbelgetrieben. Ing.-Arch. 18, 167–177 (1950).

The relative velocities of the driving crank and the driven crank of simple and compound four-bar linkages are expressed analytically in terms of the parameters, namely, the ratios of the lengths of the cranks and the connecting-rods to the length of the fixed bar. If a slide connection is used, the eccentricity of the slot furnishes another parameter. The various types of four-bar mechanisms (such as the oscillating crank type and the cross-bar type) are defined by certain inequalities among these parameters. Curves of the relative velocities for special values of the parameters are given. For a special compound four-bar linkage, these curves are resolved into a Fourier series and the Fourier coefficients are plotted as families of curves up to the sixth term. The use of these curves facilitates the selection of the parameters to give an approximation to a desired motion. *M. Goldberg* (Washington, D. C.).

Wunderlich, W. Höhere Radlinien als Näherungskurven. Österreich. Ing.-Arch. 4, 3–11 (1950).

Als "Radlinie ster Stufe" wird die Bahnkurve des Endpunktes P eines s-gliedrigen Gelenkpolygons bezeichnet, dessen Anfangspunkt O fest ist und dessen Glieder sich mit den konstanten (absoluten) Winkelgeschwindigkeiten $\omega_1, \omega_2, \cdots, \omega_s$ drehen. Die von P durchlaufene Radlinie lässt sich mittels des reellen Zeitparameters t durch $z = \sum_{r=1}^s a_r e^{i\omega_r t}$ beschreiben. In der vorliegenden Arbeit wird gezeigt, wie eine vorgelegte geschlossene Linie mit zeitlich vorgeschriebener Durchlaufung durch Radlinien hinreichend höher Stufe mit beliebiger Genauigkeit angenähert werden kann. Als Beispiel wird die Approximation eines Quadrates betrachtet bei den folgenden Methoden: (a) harmonische Analyse einer bestimmten Durchlaufung (Fourier); (b) kleinste mittlere Querabweichung (Gauss); (c) stärkste Anschmiegung (Taylor); (d) kleinste Maximalabweichung (Tschebyscheff). Der Verf. vergleicht die Abweichungen für die behandelten Näherungen des Quadrates durch Radlinien 3ter Stufe. Als Beispiel einer Radlinie mit vorgeschriebenen Singularitäten (insbes. Spitzen) wird eine Hysteresisschleife durch eine Radlinie 4ter Stufe angenähert. *S. C. van Veen.*

Macmillan, R. H. The freedom of linkages. Math. Gaz. 34, 26–37 (1950).

This is an elementary derivation of Grübler's formulas for the least number of degrees of freedom f of a linkage mechanism of n links and j joints. In the plane, if j_p is the number of joints from which p links emanate, and n_p is the number of links having p joints, then

$$f = 3n - (2j_2 + 4j_3 + 6j_4 + \cdots) - 3 = 3\sum n_p - 2\sum(p-1)j_p - 3$$
$$= 2j - (n_2 + 3n_3 + 5n_4 + \cdots) - 3 = 2\sum j_p - \sum(2p-3)n_p - 3.$$

In three dimensions, if s_p is the number of screw couplings from which p links emanate (a hinge is a screw of zero pitch; a slide is a screw of infinite pitch), and $s = \sum s_p$, then

$$f = 6n - (5s_2 + 10s_3 + 15s_4 + \cdots) - 6 = 6\sum n_p - 5\sum(p-1)s_p - 6$$
$$= 5s - (4n_2 + 9n_3 + 14n_4 + \cdots) - 6 = 5\sum s_p - \sum(5p-6)n_p - 6.$$

By properly counting the multiplicities, one can reduce these formulas to those given by Grübler [see Bottema, Appl. Sci. Research A. 2, 162–164 (1950); these Rev. 11, 549]. Applications to various mechanisms are made. In particular, this paper indicates the need for removing hidden redundant constraints in engineering applications when the number of degrees of freedom exceeds f. *M. Goldberg.*

Geronimus, Ya. L. The effect of an impact on a free rigid body. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 245–252 (1950). (Russian)

Consider a body under the action of an instantaneously applied force with a given impulse. It is shown that the subsequent motion can be represented as a screw motion in terms of two point masses which are dynamically equivalent to the body. In addition, the velocity of any point along the line of impact is determined, as well as the directions of impact through any given point along which the impact will produce the maximum and the minimum velocities of the point.   *H. I. Ansoff* (Santa Monica, Calif.).

Gilvarry, J. J., Browne, S. H., and Williams, I. K. Theory of blind navigation by dynamical measurements. J. Appl. Phys. 21, 753–761 (1950).

The authors' summary is as follows: "The differential equation is considered which determines the position of a vehicle from dynamical measurements of the nongravitational acceleration **b** made internally. Three linear approximations to the gravitational field **g**(**r**) of the earth, which lead to explicit solutions of this equation, are considered and their limitations are discussed. An interval-wise solution (linear continuation) for trajectories of extended range is described, which is based on such linear approximations and has definite advantages in this application. The theory is applied to the trajectory of the German A10 vehicle."
          E. Leimanis (Vancouver, B. C.).

Vazsonyi, Andrew. Longitudinal stability of autopilot-controlled aircraft. J. Aeronaut. Sci. 17, 399–416 (1950).

The author presents the design analysis of autopilots for subsonic and supersonic aircraft. The motion of the aircraft is restricted to the so-called longitudinal motion and when the perturbations are small the aerodynamic equations of motion of the aircraft can be expressed in three ordinary linear differential equations connecting these perturbations. The exterior dynamics is characterised by determining the attitude function $\theta(t)$ for any given elevator deflection function $\delta(t)$. The equations of motion are written in such a form that by observing the equations the important dynamical features of the aircraft can be perceived. For example, the equations of motion are reduced to

$$\left[p^2+2\zeta_w\omega_w p+\tau_1\tau_2\omega_w{}^2\frac{p+2\zeta_\theta\omega_\theta p+\omega_\theta{}^2}{(\tau_1 p+1)(\tau_2 p+1)}\right]\theta+\lambda\omega_w{}^2\delta=0,$$

where $p=d/dt$ and where ω_w and ζ_w are the wind tunnel characteristics, ω_θ and ζ_θ are the phugoid characteristics, and ω_a, ζ_a, τ_1, and τ_2 are the "gyroscopic" characteristics.

As far as the interior dynamics are concerned, it is assumed that the aircraft is equipped with a direction measuring device, which senses θ the change in attitude of the aircraft. The interior dynamics then determines the elevator deflection function $\delta(t)$ for any given attitude function $\theta(t)$. For purposes of analysis and design it is necessary to break the interior dynamics into two separate components: the computer and the autopilot proper. It is assumed that a signal proportional to the change in attitude θ is fed into the computer, that the computer determines first, second, and higher order derivatives and integrals, and that a signal σ results from the computer. This signal σ excites the autopilot proper, which in turn actuates the elevators. The change in attitude θ is connected to the signal σ by a system of differential equations characterising the computer, while σ and δ are connected by a system of differential equations characterising the autopilot.

The exterior dynamics in the two cases of uncontrolled aircraft and proportional control in which δ is assumed to be a linear function of θ for both subsonic and supersonic aircraft is examined. It is found that in the case of supersonic aircraft, proportional control is unsatisfactory since it does not damp the fast oscillations that occur, and also does not steer fast enough, i.e., there is no immediate response in the angle of attack for a given elevator deflection. A study is therefore made of the appropriate control function required to stabilize the motion. This leads to a third order differential equation for θ which is studied with the aid of the cubic chart. It is found that a combination of derivative control and delayed proportional control determined by $\delta=[\mu T_A p/(1+T_a p)]\theta+[\mu/(1+\tau_a p)]\theta$, where T_A is a time constant and T_a the time lag of the autopilot, gives good steering and improves the damping of the fast oscillations. Mention is made of the design of the computer, the case of a statically unstable aircraft, aircraft controlled by rotating the wing itself, and also the aerodynamic factors that are involved in the aerodynamic coefficients, particularly as between subsonic and supersonic aircraft.
          R. M. Morris (Cardiff).

✶**Corner, J. Theory of the Interior Ballistics of Guns.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. xiii+443 pp. $8.00.

This is not merely the sole substantial recent text wholly devoted to the theory of interior ballistics, but it is practically the only book of its kind in the English language. The author emphasizes, as have many workers in the field for the last quarter century, that the complexity of the physical and chemical processes involved, and the still unsolved problems whose solutions lie in experimental research, may have rendered obsolete any refined discussion of a mathematical structure based on a few simple physical assumptions. The system of gun, projectile, and charge, while appearing to be one of the simplest of all heat engines, involves thermo-chemical changes and transient physical phenomena still calling for examination. No adequate survey of the rich contents of this excellent book is possible in the compass of a brief review. From the point of view of mathematics, perhaps the interesting features are the use of general methods of fluid mechanics, the similarity relations for a simple set of ballistic equations, the use of numerical integration and the study of first order theory and the limitations thereof. For the designer of ordnance and of ammunition, the recently enriched results concerning powder composition, granulation, leaking guns, heat transfer to the barrel, high-low-pressure guns, and tapered-bore guns, will prove of interest and value. Of course many facts of military significance remain too confidential for publication in any book available to the general public.   *A. A. Bennett.*

Hydrodynamics, Aerodynamics

Vladimirsky, Serge. Mouvement général plan non uniforme d'une plaque infiniment mince. C. R. Acad. Sci. Paris 231, 30–32 (1950).

This paper considers the fluid motion produced by a flat plate moving with given time dependent translation and rotation, while taking into account a vortex sheet formed at the trailing edge. The basic assumption, which the author considers appropriate for motions of small duration, is that the vortex sheet of unknown strength is of the same shape

as the (known) curve described by the trailing edge of the plate, and that elsewhere the fluid motion is irrotational. The author presents without proof an integral equation for the vortex strength, which is derived by adding to the above assumption the condition of finite velocity at the edges of the plate and of zero total circulation about plate and vortex sheet. Expressions for the potential and pressure are given in terms of the unknown vortex distribution.

D. Gilbarg (Bloomington, Ind.).

Vladimirsky, Serge. Sur le mouvement non stationnaire de deux plaques. C. R. Acad. Sci. Paris **230**, 1928–1930 (1950).

Hyperelliptic integrals are used to determine the complex potential of a uniform flow perturbed by the small motion of two plates (initially on a straight line parallel to the basic flow) accompanied by the production of vortex sheets at their trailing edges. From the same assumptions and basically the same methods as in the paper reviewed above, the author states that two integral equations for the unknown vortex distributions can be derived, and writes down the equations in case of pure translation of the plates.

D. Gilbarg (Bloomington, Ind.).

Vladimirsky, Serge. Mouvement différentiel non uniforme de deux plaques. C. R. Acad. Sci. Paris **231**, 211–213 (1950).

The theory of integrals of algebraic functions is applied again towards the flow problem of the preceding paper to extend the methods and results to the case of differential motion of the two plates, in particular, to the case when one of the plates is at rest and a segment of the second plate rotates about a fixed point.   *D. Gilbarg.*

Dolaptschiew, Bl. Zweiparametrige Wirbelstrassen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 287–320 (1943). (Bulgarian. German summary)

Dolaptschijew, Bl. Über die schräge Fortbewegung der Wirbelstrassen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 137–163 (1947). (Bulgarian. German summary)
[Volume number misprinted 42 on title page.]

Dolaptschijew, Bl. Über die Stabilisierung der Wirbelstrassen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 165–178 (1947). (Bulgarian. German summary)
[Volume number misprinted 42 on title page.]

Gherardelli, Luigi. Sull'equazione del moto gradualmente vario, o lineare. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11**(80) (1947), 76–84 (1949).

The theory for two-dimensional flow of water in open channels is developed on the basis of the usual assumption that the depth of the water is small enough to permit neglect of the terms in the Euler equations containing the vertical component of the velocity. The author is interested in developing a more accurate treatment of the resistance coefficients than is customary, but he comes to the conclusion that the resulting corrections are in general not large enough to have practical significance.   *J. J. Stoker.*

Biesel, F. Étude théorique de la houle en eau courante. Houille Blanche **5**, 279–285 (1950).

The purpose of this paper is to study the speed of propagation of surface gravity waves of small amplitude in water of constant finite depth. It is, however, not assumed that the velocity of the water particles is small, but rather that the wave motion takes place relative to a prescribed flow in the horizontal direction. Various special cases are worked out in which the horizontal velocity component of the prescribed flow varies linearly in the depth coordinate.

J. J. Stoker (New York, N. Y.).

John, Fritz. On the motion of floating bodies. II. Simple harmonic motions. Comm. Pure Appl. Math. **3**, 45–101 (1950).

Although this paper is a continuation of an earlier one [same journal **2**, 13–57 (1949); these Rev. **11**, 279] it can be read with little reference to the first part. It is concerned with the harmonic motion caused by a bounded rigid body which either floats freely or is subject to forced harmonic oscillations. The linearized free surface condition and small motions of the body are assumed; for most of the work the bottom is taken to be horizontal or infinitely deep. By introduction of a suitable Green's function and use of Green's formula, the complex velocity potential for the motion can be expressed as a sum of an everywhere regular function (corresponding to the wave motion without the body, the "primary wave motion") and a function (corresponding to the waves caused by the body) satisfying a weakened form of the Sommerfeld radiation condition at infinity. The behavior and form of these two functions is studied in some detail. The uniqueness of the motion is considered next. In the case of forced oscillation it is shown that the motion is uniquely determined by the primary wave motion and the motion of the body, provided that no vertical line through the body passes through the free surface; in the case of the freely floating body this is proved only when the frequency is sufficiently high. The existence of the motion for a given primary wave is reduced to establishing the existence of a solution for a Fredholm integral equation of the second kind. This is carried out in the case of forced oscillation of a body satisfying the condition above and intersecting the free surface perpendicularly.

J. V. Wehausen (Providence, R. I.).

Couchet, Gérard. Compléments à propos des mouvements plans à circulation constante. C. R. Acad. Sci. Paris **231**, 112–114 (1950).

In an earlier note [same C. R. **221**, 280–282 (1945); these Rev. **7**, 343] the author has considered those problems of two-dimensional nonuniform motion of airfoils in incompressible fluids for which the circulation around the airfoil remains constant. In the present note he gives, without proof, formulas concerning the effect of a finite number of free concentrated vortices in the flow field.

E. Reissner (Cambridge, Mass.).

Couchet, Gérard. Les mouvements plans non stationnaires a circulation constante et les mouvements infiniment voisins. (Aile d'allongement infini.) O.N.E.R.A. Publ. no. 31, iv+79 pp. (1949).

Fogarty, L. E., and Sears, W. R. Potential flow around a rotating, advancing cylindrical blade. J. Aeronaut. Sci. **17**, 599 (1950).

Extension of the results in a paper by Sears [same vol., 183–184 (1950); these Rev. **11**, 623] to include circulation and motion in the direction of the axis of rotation.

DeYoung, John. Theoretical antisymmetric span loading for wings of arbitrary plan form at subsonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2140**, ii+95 pp. (1950).

An approach to the problem of antisymmetric loading is made by means of a lifting surface theory in which the usual lifting surface is replaced by a lifting vortex along the wing one-quarter-chord line. The boundary condition employed specifies that there be no flow through the lifting surface at the three-quarter-chord line. The method takes into account the effects of compressibility and spanwise variation of the section lift-curve slope. An antisymmetric loading is encountered, for example, in determining the aerodynamic effects due to a rolling wing, to aileron deflection, to sideslip of wings with dihedral. As an application of the method, the loading is calculated (at three points within the wing span) and the aerodynamic characteristics determined for the straight-tapered wing with reference to the three problems mentioned. Good agreement with experimental results, in general, is obtained. Extensive charts are furnished to permit rapid calculation of the loading. The example of the straight-tapered wing is examined in some detail with respect to effects of plan-form parameters on aerodynamic characteristics.   *E. N. Nilson* (Hartford, Conn.).

Perl, William. Calculation of transonic flows past thin airfoils by an integral method. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2130**, i+96 pp. (1950).

A method of calculating two-dimensional compressible flows past thin airfoils in the transonic speed range is presented. Starting from the equation of continuity in integral form, the conservation of energy in the integrated form of Bernoulli's equation, and the irrotationality condition in terms of the stream line curvature, the analysis is developed in terms of this stream line curvature which is expressed as a function of the velocity v of the fluid and the chordwise location x. Thence integral expressions for the airfoil velocity V and the local velocity v at lateral distance y are determined in terms of a streamline curvature function. By an appropriate choice of curvature function the flow pattern can be calculated.

The small perturbation form of the equations is then developed and the subsonic or Prandtl-Glauert range for the free-stream Mach number M_0 and the lower transonic range in which M_0 is increased towards unity are considered. The results conform with transonic similarity theory and by a specific choice of curvature function an explicit solution is found in the small perturbation case. The results lead to the question of the potential limit phenomenon and make it easy to understand why a subsonic free-stream speed exists for a given aerodynamic shape such that continuous symmetric type flow patterns containing local supersonic regions can, at least in principle, be derived for this shape, below this particular speed but not above. The free stream Mach number corresponding to this particular speed is called the potential limit Mach number $M_{0,l}$ and, as the free stream Mach number M_0 increases beyond this value $M_{0,l}$, it is shown that two flow patterns are theoretically derivable. One is symmetric, being a physically unreal solution containing forbidden regions bounded by limiting lines, and the other is a physically real asymmetric solution containing an asymmetric locally supersonic region. Termination of this locally supersonic region on the downstream side by a shock requires additional hypotheses concerning the shock.

These hypotheses are quite unrelated to the limiting line phenomenon.

This terminal shock in the lower transonic speed range $M_{0,l} < M_0 < 1$ is then discussed in some detail, and the resulting asymmetric flow patterns derived. The terminal shock, which is assumed to be normal, forms the boundary of the local supersonic region and moves towards the trailing edge as M_0 increases towards unity. The velocities across the shock are assumed to be related by Prandtl's equation, derivable from the Rankine-Hugoniot relations, which take a special form in the small perturbation case, and an additional assumption is made that the pressure in the downstream of the shock equals the free stream pressure. Other shock conditions are mentioned and it is realised that the calculations are of a provisional and exploratory character, but it is demonstrated that the methods used can yield results of the right order of magnitude.

Finally the upper transonic or detached shock regime which extends from the free stream Mach number unity to that supersonic Mach number $M_{0,a}$ at which the shock ahead of the airfoil becomes attached to the leading edge, assumed sharp-edged, is dealt with briefly. As in the preceding section a Prandtl-Meyer velocity distribution on the airfoil is assumed in the supersonic part and thence expressions for the velocity distributions and distance of the normal front portion of the shock from the leading edge of the airfoil are derived in terms of the free stream Mach number and this velocity distribution on the airfoil. The pressure drag for symmetrical airfoils at zero lift is calculated by means of the momentum integral. Comparisons are made throughout with experiment and other theories and the main results of the method agree qualitatively with experimental data available.   *R. M. Morris* (Cardiff).

Sauter, Fritz. Der flüssige Halbraum bei einer mechanischen Beeinflussung seiner Oberfläche. (Zweidimensionales Problem). Z. Angew. Math. Mech. **30**, 149–153 (1950). (German. English, French, and Russian summaries)

'i he author considers the propagation of sound waves in two spatial dimensions and time. By use of Fourier integrals, the author obtains a solution of the above problem in terms of an "influence function" (or a resolving kernel). This kernel is expressed as a multiple integral of the "elementary sources" and the author's problem is to furnish an explicit evaluation of this kernel. This is accomplished by use of contour integrals. The general theory contained in the paper is well known. However, some of the details may be of value to those interested in linearized nonsteady flows of a compressible fluid.   *N. Coburn* (Ann Arbor, Mich.).

Martin, M. H. Plane rotational Prandtl-Meyer flows. J. Math. Physics **29**, 76–89 (1950).

The author calls a steady plane flow of a nonviscous, nonthermally conducting fluid a general Prandtl-Meyer (GPM) flow if its hodograph degenerates to a curve, $\theta = \theta(q)$, and if the flow is neither an irrotational Prandtl-Meyer flow nor one whose streamlines are parallel straight lines or concentric circles. By considering the equations of motion in the independent variables p, ψ (p = pressure, ψ = stream function), the author arrives at a characterization of the GPM flows which correspond to a prescribed Bernoulli function $q = q(p, \psi)$ (or, equivalently, to prescribed energy and entropy distributions as functions of ψ); namely,

a curve $\theta=\theta(q)$ is the hodograph of a GPM flow if and only if there holds an identity of the form (1) $\theta''-a\theta'-b\theta'^2=0$, where $a=a(p,\psi)$, $b=b(p,\psi)$ are certain known functions when $q(p,\psi)$ is given. Two possibilities are distinguished according as (A) a, b are functions of q alone, in which case (1) is an ordinary differential equation of which every solution defines the hodograph of a GPM flow, or (B) a, b are not dependent on q alone, in which case $\theta(q)$ is uniquely determined to within sign and an additive constant. When this result is applied to the isoenergetic, anisentropic flows of a gas with separable equation of state, $\rho=\sigma(S)\pi(p)$ ($S=$entropy), it follows that the only GPM flows of type (A) are flows of polytropic gases with one of two special entropy distributions, and conversely, that the only isoenergetic, anisentropic GPM flows of a polytropic gas are of type (A). The author concludes by noting several differences between the GPM flows and the irrotational Prandtl-Meyer flows, among them the fact that for rotational flows the characteristics in the hodograph plane do not coincide with the hodographs of the GPM flows.   *D. Gilbarg.*

Seeger, R. J., and Polachek, H. On shock wave phenomena: Waterlike substances. Symposium on shock-wave phenomena, 30 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1135**, pp. 37–82 (1950).

A waterlike substance is one for which the specific internal energy $E(\rho,S)$ ($\rho=$density, $S=$entropy) is of the form $E(\rho,S)=E_1(\rho)+E_2(S)$. This paper extends the known results concerning shock wave phenomena for gases, particularly shock wave interactions, to these substances. The details in most part parallel those for gases [for which see, e.g., Courant and Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948; these Rev. **10**, 637; Polachek and Seeger, Proc. Symposia Appl. Math., v. 1, pp. 119–144, Amer. Math. Soc., New York, 1949; these Rev. **10**, 758]. Certain differences between shock wave interactions in waterlike substances and ideal gases are discussed, e.g., in the phenomena of a shock or rarefaction wave overtaking a shock, and in the conditions for existence of three-shock solutions.   *D. Gilbarg* (Bloomington, Ind.).

Nicolas, J., et Audic, H. Tableaux de calculs et diagrammes relatifs aux ondes de choc obliques en écoulements bi-dimensionnels. O.N.E.R.A. Publ. no. 11, iv+13 pp. (5 plates) (1948).

Nicolas, Jacques. Note sur une méthode de tracé d'écoulements supersoniques à deux dimensions. O.N.E.R.A. Publ. no. 12, 10 pp. (3 plates) (1948).

Ray, M. Linearized supersonic flows around a body of revolution. Bull. Calcutta Math. Soc. **42**, 31–36 (1950).

If the flow from a source distribution of constant intensity on a straight line segment OO' and a uniform supersonic flow parallel to OO' are superposed, the equations of the streamlines can be integrated in elementary terms. For several free stream Mach numbers and source strengths the author has calculated shapes of streamlines to be interpreted as meridians of blunt-nosed bodies. It should be mentioned that all streamlines emanating from points of OO' between O and the noses of these objects meet OO' and the Mach cone with vertex O at right angles, and that the velocity becomes infinite at this Mach cone and on OO'.

J. H. Giese (Havre de Grace, Md.).

Strang, W. J. Transient source, doublet and vortex solutions of the linearized equations of supersonic flow. Proc. Roy. Soc. London. Ser. A. **202**, 40–53 (1950).

Continuing his study [same Proc. Ser. A. **195**, 245–264 (1948); these Rev. **10**, 411] of singular and discontinuous solutions of the linearized potential equation of unsteady flow, the author considers first stationary point sources and line sources of finite length, whose strengths vary as the unit-step function $H(t)$, t denoting time. Next, he treats moving point sources and "needle sources," i.e., lines of such sources moving in the direction of the line, again emitting in proportion to $H(t)$. He then investigates a "gust source," which is a line of sources moving in the same direction, which begin to emit as they cross a fixed front. Also considered are some analogous doublet-type solutions. Finally, a certain generalization of ordinary hydrodynamical vortex flow (plane) is found by distributing stationary doublets of the $H(t)$-type uniformly over a half-plane. The solution, which depends on polar coordinates taken around the edge of the half-plane, coincides with vortex flow for radii small compared with at, a being the speed of sound, but for larger radii varies widely with the angular coordinate.

W. R. Sears (Ithaca, N. Y.).

Strang, W. J. Transient lift of three-dimensional purely supersonic wings. Proc. Roy. Soc. London. Ser. A. **202**, 54–80 (1950).

Here the results of the paper reviewed above are applied to the calculation of lift, pitching moment, and pressure distribution of wings having supersonic edges. The "needle" and "gust" sources are most useful, giving by straightforward integration the cases of sudden change of incidence and entry into a sharp-edged gust. In both cases the elementary wing treated is a triangular half-wing, i.e., a wing whose planform consists of that portion of the plane $z=0$ bounded by the lines $y=0$ and $y=x\tan\gamma>x\tan\mu$. Here the x direction is the stream direction and μ is the Mach angle. Any supersonic planform having a contour made up of a finite number of straight lines can be constructed of these half-wings by superposition. It is found that the pressure-growth functions for the various significant regions in both cases can all be expressed in terms of a function $P(\alpha,\beta)=\pi^{-1}\cos^{-1}\{(\alpha-\beta)/(1-\alpha\beta)\}$. These results are collected, and from them the respective lift- and moment-growth functions are calculated and are plotted for typical cases. The results are related to steady-flow and two-dimensional results in the appropriate limiting cases.

W. R. Sears (Ithaca, N. Y.).

Tournier, Marcel, et Bassière, Marc. Les mouvements tourbillonnaires en régime transitoire. O.N.E.R.A. Publ. no. 16, vi+42 pp. (1948).

Krasil'ščikova, E. A. On the theory of the unsteady motion of a compressible fluid. Doklady Akad. Nauk SSSR (N.S.) **72**, 23–26 (1950). (Russian)

Consider a thin vibrating wing translated parallel to the x-axis with constant supersonic speed u at a small angle of attack. The velocity potential $\phi(x,y,z,t)$ is assumed to satisfy the linearized equation

(i) $(a^2-u^2)\phi_{xx}+a^2(\phi_{yy}+\phi_{zz})-\phi_{tt}-2u\phi_{tx}=0,$

referred to axes translated with velocity $u,0,0$. Let P be the projection of the wing onto $z=0$. On P the normal component of velocity is prescribed to be

$v_z=\partial\phi/\partial z=A(x,y)f(t+\alpha(x,y)).$

Then

$$\phi = \int\int_S C(\xi, \eta) K(x, y, z, t; \xi, \eta) d\eta d\xi,$$

where K is the fundamental solution of (i) defined by $Kr = \sum_{j=1}^{2} f\{t + \alpha(\xi, \eta) - (ux - u\xi + (-1)^j iar)/(u^2 - a^2)\}$, where $r^2 = (x-\xi)^2 - k^2(y-\eta)^2 - k^2z^2$, $k^2 = (u/a)^2 - 1$, and $S(x, y, z)$ is the intersection of $z = 0$ and the Mach forecone of x, y, z. At points of P, $C(x, y) = -\frac{1}{2}\pi A(x, y)$. Outside P and the range of influence of the trailing vortex system, C is defined by the integral equation

$$(ii) \quad \int_{x'}^{x'} \int_{\psi(l')}^{y'} C(\xi', \eta') K(x, y, 0, t; \xi, \eta) d\eta' d\xi' = F(x', y'),$$

where $x' = x - ky$, $y' = x + ky$, $\xi' = \xi - k\eta$, $\eta' = \xi + k\eta$; $x - ky = x'_z$ is tangent to the leading edge $y' = \psi(x')$; and $F(x', y')$ is a known function. For $v_n = \Re A_2(x, y) \exp(i\omega t)$, (ii) reduces to an integral equation previously obtained and solved by the author for harmonic vibrations [same Doklady 56, 571–574 (1947); Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 147–164 (1947); these Rev. 9, 392].

J. H. Giese (Havre de Grace, Md.).

Miles, John W. **The oscillating rectangular airfoil at supersonic speeds.** Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-15**, ii+54 pp. (1949).

One of the unsolved problems in supersonic aerodynamics is the linearized unsteady flow about an oscillating rectangular thin wing. Owing to the disturbance always propagating within the Mach cone, the problem can be reduced to the basic problem of the aerodynamic behavior of a quarter infinite wing with assigned, time-dependent wing slope. To treat this problem the author extends the method of the double Fourier transform which is well known in electromagnetic theory and quantum mechanics. The solution is effected with the aid of the Wiener-Hopf technique on singular differential equations and yields a Green's function which may be expressed as a finite integral or as a series. It can be applied to calculate aerodynamic behavior for effective aspect ratio (AR $(M^2 - 1)^{\frac{1}{2}}$) greater than unity. The author explains clearly the restrictions of the two spectrum parameters in the Fourier transform in order to obtain convergent solutions. This is particularly delicate in the problem of wave propagation where an ordinary discontinuity in the function is allowable. The author also shows that the force and moment coefficients of practical interest may be expressed in terms of known functions including certain integrals which have been calculated for the two-dimensional oscillating wing. It may be pointed out that the author uses "airfoil" to mean "wing" which is commonly reserved for a three-dimensional lift surface.

C. C. Chang (Baltimore, Md.).

Miles, John W. **The indicial admittance of a supersonic rectangular airfoil.** Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-27**, i+27 pp. (1949).

As a useful example to demonstrate the author's work reviewed above, this paper gives the aerodynamic response of a supersonic rectangular wing under two loading conditions: (1) under a suddenly changing angle of attack defined by an assigned unit step function of time; (2) under the influence of a sharp edged gust. He breaks down both the lift and moment coefficients into two parts: One is two-dimensional in nature and the other is due to interaction of the tip. The results are valid only for such wing

planforms that the Mach lines from the leading edge corners do not intersect the opposite side edges.   *C. C. Chang.*

Miles, John W. **Transient loading of wide delta airfoils at supersonic speeds.** Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-37**, i+50 pp. (1950).

As a further application of the author's earlier work [see the second preceding review], a delta wing with supersonic loading edge is treated under various unsteady conditions: (a) sudden change of angle of attack; (b) different types of gust. Results are given in graphical form.   *C. C. Chang.*

Dörr, J. **Les forces aérodynamiques sur une aile vibrant harmoniquement dans un écoulement supersonique.** O.N.E.R.A. Publ. no. 37, iv+29 pp. (1949).

Dörr, J. **Détermination des forces aérodynamiques instationnaires. Système plan, fluide incompressible. Méthode Betz—L. Schwarz et méthode Küssner.** O.N.E.R.A. Publ. no. 9, 29 pp. (1948).

von Schwarz, M. J. **Détermination des forces aérodynamiques instationnaires. Système plan, fluide incompressible. Méthode de Jaeckel.** O.N.E.R.A. Publ. no. 6, 41 pp. (1948).

de Schwarz, M. J. **Application de la méthode du potentiel d'accélération au calcul des forces aérodynamiques instationnaires en régime supersonique. Problème plan.** O.N.E.R.A. Publ. no. 25, iv+31 pp. (1949).

Weber, R. **Détermination des coefficients aérodynamiques instationnaires en régime supersonique. Problème plan, méthode L. Schwarz.** O.N.E.R.A. Publ. no. 5, 69 pp. (1948).

Weber, R. **Table des coefficients aérodynamiques instationnaires. Régime plan supersonique. I. Exposé sommaire de la méthode de L. Schwartz et introduction aux tables.** O.N.E.R.A. Publ. no. 41, iv+26 pp. (1950).

Weber, R. **Tables des coefficients aérodynamiques instationnaires. Régime plan supersonique. II. Tables numériques des coefficients aérodynamiques instationnaires.** O.N.E.R.A. Publ. no. 41, 98 pp. (1950).

Weber, R. **Tables des coefficients aérodynamiques instationnaires. Régime plan supersonique. III. Représentations graphiques coefficients aérodynamiques instationnaires et tables de Küssner-Jordan.** O.N.E.R.A. Publ. no. 41, i+134 pp. (1950).

Viaud, Louis. **Contribution à l'étude des écoulements supersoniques à deux dimensions. Méthode graphique de tracé d'écoulements comportant des ondes de choc et des détentes.** O.N.E.R.A. Publ. no. 13, iv+14 pp. (5 plates) (1948).

Wannier, Gregory H. **A contribution to the hydrodynamics of lubrication.** Quart. Appl. Math. 8, 1–32 (1950).

This paper develops in further mathematical detail the problem of lubrication by applying the Stokes, rather than Reynolds, equation in order to take into account the three-dimensional effects. To demonstrate the advantage of the new approach, the author is able to show that, by assuming the liquid film to be thin, the Reynolds equation corresponds

to the first approximation of the Stokes equations but that in special situations the Reynolds equation is unable to distinguish cases where the flow is from high to low pressure from those where the direction of flow is reversed. As examples, the nonconcentric cylindrical bearing and nonconcentric spherical bearing are treated. The discussions of the results appear to be quite exhaustive. Both the streamlines and the constant pressure lines are traced. In the case of the cylindrical bearing, the load versus eccentricity is compared with that obtained by Sommerfeld. The author contends that the classical theory on lubrication proceeds not from Stokes equations but from Reynolds equation. This seems to overlook the work of certain earlier authors such as Joukowsky [Communications et procès-verbaux de la société mathématique de Kharkow **1887**, 31–46 (1887)], S. C. Mitra [Bull. Calcutta Math. Soc. **14**, 151–160 (1924)], and G. B. Jeffrey [Proc. London Math. Soc. (2) **14**, 327–338 (1915)]. The last named, particularly, has solved both problems considered by the present author.   *Y. H. Kuo.*

Lighthill, M. J. Contributions to the theory of heat transfer through a laminar boundary layer. Proc. Roy. Soc. London. Ser. A. **202**, 359–377 (1950).

An approximation to the heat transfer rate across a laminar incompressible boundary layer, for arbitrary distribution of main stream velocity and of wall temperature, is obtained by using the energy equation in von Mises' form, and approximating the fluid coefficients in a manner which is most closely correct near the surface. This approximation is in the form of an integral expression involving the local skin friction coefficient and the excess of wall temperatures over main stream temperature. The formula is obtained by an asymptotic analysis for large Prandtl numbers but correlation with semi-empirical formulae indicates satisfactory confirmation in form for Prandtl numbers as low as 0.7. The same analysis is shown to be applicable at high Mach numbers so that the results may be applied to the problem of equilibrium between aerodynamic heating and radiation. This is considered in particular for the nonuniform case of temperature development near the nose of a body moving at high velocities. This latter phase of the analysis requires the solution by series approximation of a nonlinear integral equation.   *N. A. Hall* (Minneapolis, Minn.).

Monin, A. S. Dynamic turbulence in the atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 232–254 (1950). (Russian)

In the first part of this paper the author derives equations for the rate of change of turbulent energy and of the mean velocity components under hypotheses concerning the Reynolds' stresses and the diffusion of turbulent energy. Further similarity hypotheses and the logarithmic velocity distribution law are assumed and in the second part of the paper the resulting equations are used to study the structure of the wind over the Earth. Tables and graphs are computed for various quantities of meteorological interest.

J. V. Wehausen (Providence, R. I.).

Čarnyǐ, I. A. On a grapho-analytic method in the theory of filtration. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 961–965 (1950). (Russian)

The region of flow to be studied, lying between two surfaces of equal pressure, H_1 and H_2, is divided into approximating stream tubes of cross-section $\sigma(S)$ depending on the coordinate S measured along the stream tube. Though pro-

vision is made for a more general law of filtration, all examples and most discussion relates to D'Arcy's law, $q = -c\sigma dH/dS$, where q is the flux. For incompressible liquids this gives, on integration, $q = cR^{-1}(H_1 - H_2)$, where the numerically computable $R = \int_{H_1}^{H_2}\sigma^{-1}dS$ is called the filtration resistance of the tube. The total filtration resistance per unit length $f = [\sum_{i=1}^{n}R_i^{-1}]^{-1}$ for an eccentrically placed tile line within a larger tile is computed with $n=10$ and with wedge-shaped stream tubes. Comparison with the analytically determined value shows best agreement when eccentricity is small, poorer agreement for larger eccentricities.   *R. E. Gaskell* (Ames, Iowa).

Zilsel, P. R. Liquid helium II. The hydrodynamics of the two-fluid model. Physical Rev. (2) **79**, 309–313 (1950).

Nonlinear equations of motion for reversible processes in the two-fluid model of liquid helium II are derived from a variation principle of a type first introduced by Eckart [same Rev. (2) **54**, 920–923 (1938)]. The possibility of transitions between the two fluids, disregarded so far, has been taken into account. The theory leads to a new equilibrium condition between the two fluids depending on their relative velocity.   *F. London* (Durham, N. C.).

Elasticity, Plasticity

Green, A. E., and Shield, R. T. Finite elastic deformation of incompressible isotropic bodies. Proc. Roy. Soc. London. Ser. A. **202**, 407–419 (1950).

The authors recapitulate their tensorial treatment of the classical theory of finite strains [Philos. Mag. (7) **41**, 313–336 (1950); these Rev. **11**, 627], hence obtaining some equations employed by Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 379–397 (1948); these Rev. **10**, 340], equations which except for the addition of a hydrostatic pressure are identical with special cases of results of Finger [Akad. Wiss. Wien, S.-B. IIa. **103**, 1073–1100 (1894), equation (35)] and of Almansi [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **20**₁, 705–714 (1911)]. The authors give a short and efficient derivation of Rivlin's general solution [Proc. Cambridge Philos. Soc. **45**, 485–487 (1949); these Rev. **10**, 650] of the torsion problem for an incompressible circular cylinder. The paper closes with two original contributions, both being exact solutions of the finite elastic strain equations for an incompressible material with arbitrary strain energy. In the first, a circular cylinder is rotated about its axis at uniform angular speed. As other solutions of this type have taught us to expect, in order to keep the curved mantle free of tractions, it is necessary to supply a suitable distribution of normal stress on the plane ends. Unfortunately, the present solution contains an arbitrary parameter to which it is difficult to assign a meaning. If the resultant force on the plane ends is to be zero, this parameter must satisfy a polynomial equation, the nature of whose roots the authors do not attempt to discuss even for special forms of the strain energy. The second solution is appropriate to a spherical shell subject to uniform internal and external pressures. This problem of considerable difficulty is solved, and the results are specialized to the case of a spherical cavity in an infinite body and to the case of a thin spherical shell. For a thin shell of the simplest type of incompressible body (Rivlin's "neo-Hookean material") the difference between outside and inside pressures is greatest

when the shell radius has been increased by about 40%; in a Mooney material the pressure difference is everywhere somewhat greater than in a neo-Hookean material.

C. Truesdell (Bloomington, Ind.).

Signorini, Antonio. Un semplice esempio di "incompatibilità" tra la elastostatica classica e la teoria delle deformazioni elastiche finite. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 276–281 (1950).

Some time ago [Atti della Società Italiana per il Progresso delle Scienze, 24 Riunione, Palermo, 1935, v. 3, pp. 6–25 (1936)] the author announced the following result [cf. also Ann. Mat. Pura. Appl. (4) **30**, 1–72 (1949), chap. I, § 3; these Rev. **11**, 756; and C. Tolotti, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1139–1162 (1942) = Ist. Naz. Appl. Calcolo (2), no. **145**; these Rev. **7**, 141]. In the general equations of finite elastic strain, let the displacements and the applied loads be expanded formally in power series in an arbitrary parameter. The vanishing of the terms of lowest degree then yields the equations of the linear theory. But there are cases when corresponding to a given solution in the linear theory the conditions of integrability for the terms of next highest order in the power series are not satisfied, and thus the formal power series solution of the equations of the general theory fails to exist. This situation the author calls "incompatibility," and in the present note he shows that the case $t_{xx}=ay$, $t_{yy}=bx$, $ab\neq0$, while the remaining stress components vanish, is a case of incompatibility in this sense. The author regards incompatibility as indicating need for modifying conventional views regarding elasticity theory, and in particular he claims that the complete indeterminacy of the rotation must be relinquished in these cases. [While the results obtained are of interest in themselves, the reviewer is unable to see that from them follow any general conclusions regarding the classical linear theory of elasticity, since there is no reason whatever to expect a classical solution to be the first term in a power series solution of the general equations. Rather, since the differential equations of even a second order theory are of a higher degree than those of the linear theory, one would expect the classical solutions to be in a possibly quite elaborate asymptotic relation to solutions of the general theory.] *C. Truesdell* (Bloomington, Ind.).

Lévi, Franco. Étude directe des équilibres élastiques en présence de déformations non compatibles. C. R. Acad. Sci. Paris **231**, 209–211 (1950).

The author considers a plane elastic system subject to initial stress. It was shown by Colonnetti [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **27₂**, 267–270, 331–335 (1918)] that the corresponding strains ∂_{ij} (which need not satisfy the conditions of compatibility) must satisfy a certain variational principle. By introducing the Airy stress function F and writing out the Euler equation of this variation, the author obtains the differential equation $\nabla^4 F + E(\partial_{yy,xx} - 2\partial_{xy,xy} + \partial_{xx,yy}) = 0$, where E is Young's modulus. *C. Truesdell* (Bloomington, Ind.).

Kammerer, Albert. Les écarts à la loi de Hooke et le domaine de stabilité. C. R. Acad. Sci. Paris **231**, 681–683 (1950).

The paper deals with a homogeneous elastic body for which the principal stresses σ_i are quadratic functions of the principal strains e_i. The Jacobian of the transformation from σ_i to e_i is Δ. The stability condition is then $\Delta = 0$.

In the σ_i space, this condition defines a surface which divides the space into several regions one of which gives stability. The geometry of this surface is considered in some special cases. *G. E. Hay* (Ann Arbor, Mich.).

Mindlin, Raymond D., and Cooper, Hilda L. Thermoelastic stress around a cylindrical inclusion of elliptic cross section. J. Appl. Mech. **17**, 265–268 (1950).

The authors derive closed expressions for the stress distribution in an unbounded medium containing a cylindrical inclusion of elliptic cross section for the case of a uniform change of temperature. The previous work of Donnell [Theodore von Kármán Anniversary Volume, pp. 293–309, California Institute of Technology, Pasadena, Calif., 1941; these Rev. **3**, 30] is cited by the authors as suggesting the existence of such a simple solution to the problem in question. The paper contains detailed computations of the technically significant components of stress at the interface between the two media. *A. W. Saéns.*

Storchi, Edoardo. Integrazione delle equazioni indefinite della statica dei veli tesi su una generica superficie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 326–331 (1950).

Airy, Finzi [same Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584, 620–623 (1934)], and the author [ibid. (8) **7**, 227–231 (1949); (8) **8**, 116–120 (1950); these Rev. **11**, 556, 757] have obtained general solutions of the equilibrium equations for two-dimensional continua for the case of the plane, developable surface, surface of constant curvature, and surface of revolution. The author now obtains a general solution for a surface of any form, to this end employing geodesic orthogonal coordinates. His result, too elaborate to recapitulate here, gives the three stress components in terms of the values of an arbitrary function and of its first five derivatives. He notes an exceptional class of surfaces for which the fifth derivatives do not appear.

C. Truesdell (Bloomington, Ind.).

Ilieff, Lübomir. Einige Probleme über nichtgleichmässig gespannte ebene Membranen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 251–286 (1943). (Bulgarian. German summary)

Djakov, E., und Christov, Chr. Bemerkungen zu der Arbeit "Einige Probleme ueber nichtgleichmässig gespannte ebene Membranen" von L. Ilieff. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 427–429 (1943). (Bulgarian. German summary)

See the preceding title.

Neuber, H. Allgemeine Schalentheorie. I. Z. Angew. Math. Mech. **29**, 97–108 (1949). (German. Russian summary)

Neuber, H. Allgemeine Schalentheorie. II. Z. Angew. Math. Mech. **29**, 142–146 (1949).

The author presents an approach to the theory of thin elastic shells, based on developments in powers of a thickness coordinate of the solutions of the differential equations of a three-dimensional isotropic elastic continuum. The notation is that of tensor analysis. [The theory is not "allgemein" but rather the linearised theory. Contrary to the author's belief his developments do not permit an arbitrarily close approximation to the results of an exact three-dimensional theory since he makes use of the classical hypothesis that normals to the undeformed middle surface are deformed,

without extension, into normals to the deformed middle surface.]   *E. Reissner* (Cambridge, Mass.).

Zanaboni, Osvaldo. Nuovi punti di vista sulle condizioni ai limiti delle lastre sottili. Ann. Triestini. Sez. 2. (4) 1(17), 53–72 (1947).

The author's objective is to clarify the meaning of Kirchhoff's boundary conditions for elastic plates. By the laws of statics he formulates the principle of virtual work directly in terms of resultants. By varying all five of these independently he obtains the Poisson boundary conditions; by requiring that the virtual work of the normal cross-resultant be zero, thus ankylosing one internal degree of freedom, he obtains the Kirchhoff boundary conditions. While indeed, as the author remarks, his results are purely statical and do not depend upon Hooke's law or upon assumptions regarding the deformation, the same may be said also of the more vivid and illuminating statical explanation given by Kelvin and Tait [cf., e.g., Love, A Treatise on the Mathematical Theory of Elasticity, Cambridge University Press, 4th ed., 1927, § 297].   *C. Truesdell* (Bloomington, Ind.).

Zanaboni, Osvaldo. Azioni interne e deformazioni intorno ad un punto nelle lastre a doppia curvatura. Ann. Triestini. Sez. 2. (4) 1(17), 73–83 (1947).

From statical considerations the author obtains the laws of transformation for the stress and moment resultants and for the associated virtual displacements in a curved shell.
                    C. Truesdell (Bloomington, Ind.).

Nash, W. A. Bending of an elliptical plate by edge loading. J. Appl. Mech. 17, 269–274 (1950).

The author applies the known series solution of the biharmonic equation in elliptical coordinates [A. Timpe, Math. Z. 17, 189–205 (1923)] to a problem of transverse bending of thin elastic plates. The boundary conditions are (1) given deflection of the edge of the plate, (2) given bending moments along the edge. Introduction of the series solution into the boundary conditions leads to infinite systems of equations for the infinite number of coefficients in the assumed series. The paper contains explicit expressions for the coefficients of the infinite system of equations. Application to one numerical example is made.
                    E. Reissner (Cambridge, Mass.).

Girkmann, K. Berechnung eines Rohrstranges mit Gleitblechlagerung. Österreich. Ing.-Arch. 4, 115–130 (1950).

The system dealt with consists of a long cylindrical tube (pipeline), filled with liquid, the structure being supported at equidistant stations along the tube length by slide-supports. The supports are constructed so as to permit no net axial forces to develop along the tube, and each support force is assumed to consist of a uniform pressure acting over a small rectangular area. The problem is to determine the local stresses induced near the supports as a result of the weight of the liquid and tube. An approximate solution is accomplished as follows. The tube is first treated as a membrane, and imaginary rigid bulkheads are introduced at the support stations; the shear forces induced between the tube and bulkheads, due to the liquid and tube weights, is then determined. Next, these shear forces (which act around the tube periphery at the support stations) are equilibrated by the reaction loads of the supports, and the stresses in the actual, flexure-resistant tube are sought from a solution of the Flügge equations. The stresses so determined are assumed to be those actually present in the structure. By assuming the supports are sufficiently far apart, interaction between neighboring supports is neglected; hence the flexural problem is dealt with as a loading applied at a station of an infinitely long tube. The solution to the flexure problem is obtained by expressing the external loadings as the product of a Fourier series directed along the tube periphery and a Fourier integral along the (infinite) tube length. The techniques of contour integration in the complex plane are then used to evaluate the numerous integrals appearing in the formal solution. Numerical results are given for a typical system configuration.
                    M. Goland (Kansas City, Mo.).

Ōkubo, H. On the torsion of a shaft with keyways. Quart. J. Mech. Appl. Math. 3, 162–172 (1950).

The torsion problem for a circular cylinder with a number of longitudinal notches is solved mathematically in terms of the usual potential functional φ expressed in polar coordinates. In the particular case of one notch two such potential functions satisfying the required condition on the circular boundary are combined, and by means of three arbitrary constants thereby involved, the boundary of the notch, derived from the boundary condition, can be made to pass through any three chosen points. The relations between the maximum stress and the radius of curvature at the bottom of the notch are then discussed.

The problem for a circular shaft with one keyway, in a form suitable for practical use, is solved by combining potential functions of the above type in such a way that the boundary condition yields an equation for the boundary of the keyway involving seven arbitrary constants, and which can therefore be made to pass through seven chosen points. The results are compared with those obtained experimentally or numerically by previous writers and the author observes that there is considerable discrepancy between his own results and these. However, it may be noted that Parkus [Ing.-Arch. 3, 336–344 (1949); these Rev. 11, 288] pointed out the same discrepancy between his results and these same experimental results. The results of the author and Parkus for a keyway of similar relative dimensions are in comparatively close agreement.   *R. M. Morris* (Cardiff).

Reissner, H. J., and Wennagel, G. J. Torsion of noncylindrical shafts of circular cross section. J. Appl. Mech. 17, 275–282 (1950).

The authors study solutions of the equilibrium equation $\partial^2\Psi/\partial r^2 + (3/r)\partial\Psi/\partial r + \partial^2\Psi/\partial z^2 = 0$, which are of the type $r\Psi = Z_1(r) \exp z$ or $r\Psi = Z_1(ir) \cos z$, where Z_1 is a linear combination of Bessel's functions of order one and first or second type with real or imaginary argument. Here $\Psi(r, z)$ is the angle of twist of the noncylindrical shaft with circular boundaries at each section. Graphical integration is employed to find the generatrix of the shaft so that the mantel surface is free from stress. This leads to shafts of solid and hollow sections. Graphs display the twist and shearing stress $\tau_{z\theta}$ at significant sections of the shaft.   *D. L. Holl*.

Sauter, Fritz. Der elastische Halbraum bei einer mechanischen Beeinflussung seiner Oberfläche. (Zweidimensionales Problem.) Z. Angew. Math. Mech. 30, 203–215 (1950). (German. English, French, and Russian summaries)

In this paper, the author studies the propagation of the two-dimensional waves in a semi-infinite elastic solid. It is

assumed that the stresses over the plane boundary are known. The solution of this problem is stated in terms of a two-dimensional Fourier integral. By introducing shock-like initial stresses along a line and integrating in the complex plane [see the review of an earlier paper, same Z. **30**, 149–153 (1950); these Rev. **12**, 215], the author is able to express the influence functions in terms of elementary functions. The resulting influence functions are studied in detail for the problem of determining the deformations in the neighborhood of a corner. It is shown that shock-like waves are propagated.   *N. Coburn* (Ann Arbor, Mich.).

Thomson, W. T.   Matrix solution for the vibration of non-uniform beams.   J. Appl. Mech. **17**, 337–339 (1950).

The author presents an application of the matrix method to the computation of the frequencies of vibration of beams. The method differs from earlier matrix methods. It consists of the relatively simple operation of writing in matrix notation the relation between the shear, moment, slope, and deflection at the two ends of elements of the beam. These simple matrices are then combined by matrix multiplication to give a matrix relating the shear, moment, slope, and deflection at the two ends of the beam. From this matrix, one can readily derive the natural frequencies of vibration for both fixed and free end conditions. The method can be applied to both uniform and nonuniform beams. The basic assumption involved in the author's method is that the beam is of compact section, i.e., that conditions at any given station depend only on those at the immediately adjacent stations.   *S. Levy* (Washington, D. C.).

Parhomovskiĭ, Ya. M.   Properties of the forced vibrations of distributed systems with damping.   Doklady Akad. Nauk SSSR (N.S.) **72**, 651–653 (1950).   (Russian)

A bar of length L is clamped at one end ($x=0$) and a momentum $M_0 \sin \omega t$ is applied at the section $x=l$. The twisting moment at a cross section is given by

$$M = C(\partial \varphi/\partial x + (\alpha/\omega)\partial^2 \varphi/\partial x \partial t).$$

The second term accounts for the damping, C is the torsional rigidity (supposed constant along the bar), and α is the damping constant. When the square of the amplitude of the forced vibration is plotted against ω, the curve has for $\alpha \neq 0$ only a finite number of maxima (resonances) which are reached for slightly different values of ω in different sections of the bar.   *W. H. Muller* (Amsterdam).

Ashley, Holt, and Haviland, George.   Bending vibrations of a pipe line containing flowing fluid.   J. Appl. Mech. **17**, 229–232 (1950).

Aerodynamically induced vibrations have been observed in a large above-ground oil pipe line. The authors investigate these theoretically. The differential equation governing the vibration is the classical one, except that it contains an extra term due to the flow of oil. This term involves the second order mixed derivative of the deflection with respect to time and distance along the pipe. Numerical and algebraic difficulties arise when the characteristic frequencies are sought in the usual way. Accordingly, an approximate solution is obtained, in which the amplitude factor is determined as a power series in the distance x along the pipe. Numerical work then shows that under all practical circumstances the frequency of vibrations is relatively independent of the rate of flow v of the oil, while the damping increases rapidly with v, as observed in practice.   *G. E. Hay.*

Alumyaè, N. A.   Application of Castigliano's general variational principle to the investigation of the buckling stage of thin elastic shells.   Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 93–98 (1950).   (Russian)

A generalized Castigliano's principle is derived for thin elastic shells in the buckling stage under the assumption that the change in equilibrium states represents a local loss of stability. An expression for the critical loading is derived in terms of three stress functions and a work function. Approximations are shown to give both upper and lower bounds for the true loading depending on the choice of admissible functions.   *H. I. Ansoff.*

Dol'berg, M. D.   On forms of buckling of bars.   Doklady Akad. Nauk SSSR (N.S.) **71**, 839–842 (1950).   (Russian)

The author considers the buckling of a bar on many stiff supports, the influence of an additional support on the nodes, and the critical forces of the various buckling forms.

           W. H. Muller (Amsterdam).

Grammel, R.   Scherprobleme.   Ing.-Arch. **17**, 107–118 (1949).

The author considers a large number of stability problems involving shafts in torsion, tension and flexure, and circular plates under bending. In a typical problem, one end of a shaft is built-in, and a rigid circular disk is attached to the other end with its plane perpendicular to the axis of the disk. Equal and opposite forces P are then applied to the edge of the disk, acting towards each other along a diameter d. The directions and magnitudes of the two forces are fixed and the disk is rotated slightly in its original plane. This produces a stability problem involving torsion. The critical load is computed directly and also by the use of eigenvalues. In another problem, a shaft is mounted and loaded as above, but the disk is constrained to rotate about its diameter perpendicular to the diameter d. This produces a stability problem involving flexure. In still another problem a circular plate with a concentric circular hole is considered. Bending couples are applied around the edges in a manner analogous to that in which the bending couple is applied in the problem mentioned just above. This produces a stability problem involving bending of the plate.

           G. E. Hay (Ann Arbor, Mich.).

Tarabasov, N. D.   Stresses in plane elastic flush-fitted homogeneous bodies.   Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] **3**, no. 2, 3–14 (1947).   (Russian.   English summary)

Narodetsky, M. Z.   Flush fit of roller bearing rings.   Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] **3**, no. 2, 15–26 (1947).   (Russian.   English summary)

Rabinovich, A. L., and Fedotov, N. M.   Stresses in a pulley.   Engineering Rev. [Akad. Nauk SSSR. Inženernyĭ Sbornik] **3**, no. 2, 49–84 (1947).   (Russian.   English summary)

Rabotnov, Yu. N.   Some problems of the theory of creep.   Vestnik Moskov. Univ. **1948**, no. 10, 81–91 (1948).   (Russian)

Various one-dimensional engineering formulations of the creep theory are enumerated and it is pointed out that none of these provide a simultaneous description of the intimately related phenomena of creep, relaxation, plastic heredity, work hardening, and the so-called "reverse" creep. A one-dimensional theory is next formulated, as an extension of

Volterra's theory of elastic heredity, which takes all these phenomena into account. An experimental confirmation of the resulting creep equation is presented. The theory is next applied to the problems of pure bending of a thin beam and of deformation of a tube subjected to internal hydrostatic pressure. Stress distributions are computed in each case in terms of the constants associated with a creep function. An extension of the theory to three-dimensional stress distributions is made under the condition that all stresses, as well as strains, vary proportionately to the same time-dependent parameter. *H. I. Ansoff* (Santa Monica, Calif.).

Read, W. T., Jr. Stress analysis for compressible visco-elastic materials. J. Appl. Phys. 21, 671–674 (1950).

The author's summary is essentially as follows. Fourier integral methods show that static elasticity solutions can be used to determine the time-dependent stresses for any boundary conditions when stress, strain, and all their time derivatives are related by linear equations with constant coefficients. If stress and double refraction and their time derivatives are also linearly related, the standard photo-elastic techniques can be used to determine the directions and difference in magnitude of the time-dependent principal stresses, even though the principal stress axes do not coincide with the polarizing axes and both vary with time. When viscoelastic models are used in photoelastic studies, the time variation of the stress distribution in the model represents a first approximation to the dependence of the stress in the elastic prototype on Poisson's ratio. *D. C. Drucker.*

Loring, S. J. Theory of the mechanical properties of hot plastics. Trans. A.S.M.E. 72, 447–463 (1950).

This paper contains many new ideas, mathematical derivations, hypotheses, and solutions to practical problems. A treatment of large strain is given which starts from the affine deformation tensor $x_i'' = a_{ij} x_j'$ but ends with a non-tensorial representation in terms of principal extensions plus 1, α_i, only. Although it is not clear how such a description can be used in general, steady state pipe and channel shear flow problems are solved in which the velocity varies with only one coordinate. The hot plastic is assumed incompressible, and isotropic in a generalized sense. The state of stress is supposed dependent solely upon the "elastic" strain which is geometric strain plus a relaxation strain. Expressions for the strain energy function ψ and the relaxation functions R_i are developed on a simple statistical basis: $\psi = (2G/n^2)(\alpha_1^n + \alpha_2^n + \alpha_3^n)$, $R_i = -(\alpha_i/nT)(1 - \alpha_i^{-n})$. The combination of large strain elasticity and relaxation leads to an apparent viscosity for shear flows. A variation of 10 to 1 is found for several cases which is shown to agree well with experimental data. *D. C. Drucker.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Köhler, Horst. Ein einfaches Verfahren zur Ermittlung der gesamten Zerstreuungsfigur optischer Geräte aus der meridionalen Durchrechnung auf Grund der Bildfehlertheorie dritter Ordnung. Z. Angew. Math. Mech. 30, 226–228 (1950).

The author traces two rim rays and astigmatism along these rays through an optical system. He uses these results of the ray trace to obtain the coefficients of the meridional and sagittal deviations. [Reviewer's remark. Although the results of this paper are applicable in certain cases, the reviewer would like to warn against their application to systems of large aperture and field. A study of the light distribution in the image of photographic systems with an equivalent method has shown that the calculation of skew rays is indispensable for obtaining correct results, especially if the field of the instrument is sizable, since in this case the skew rays, which are not vignetted, have a large influence. The method may be sufficient for telescope lenses with small field.] *M. Herzberger* (Rochester, N. Y.).

Armsen, Paul. Über die Strahlenbrechung an einer einfachen Sammellinse. I. J. Reine Angew. Math. 187, 193–221 (1950).

This paper discusses the image formation by a lens consisting of two aspheric surfaces of rotation, the calculation being carried out to the image orders of fifth order. The paper contains real progress in the approximate design of optical systems, since the author has simplified computation by introducing a number of invariant expressions. The section on the number of image errors is historically correct, but futile, since there is no agreement on what to call an image error. In the fifth order, the image errors of all points in space depend on twelve independent data, the fifth order coefficients of the characteristic function. The errors of the points of a plane depend on nine coefficients which are independent of the surface. However, there is a possibility of reducing this number still further if one considers the fact that the correction of astigmatism alone on all rays gives sharp image formation. *M. Herzberger.*

Biot, A. Sur le calcul des véhicules redresseurs à deux lentilles identiques. Ann. Soc. Sci. Bruxelles. Sér. I. 62, 36–39 (1948).

Biot, A. Sur le condensateur à deux miroirs sphériques. Ann. Soc. Sci. Bruxelles. Sér. I. 62, 83–92 (1948).

Biot, A. Sur un problème relatif à l'astigmatisme du dioptre. Ann. Soc. Sci. Bruxelles. Sér. I. 62, 114–119 (1948).

Studies of some special problems of geometrical optics, e.g., the astigmatism of a refracting surface with the diaphragm at the aplanatic point, the study of a condensor consisting of two spherical mirrors, and the calculation of the Gaussian data of a lens system where a pair of identical achromats is put between the objective and the ocular of an optical system. *M. Herzberger* (Rochester, N. Y.).

Biot, A. Sur certains systèmes pancratiques. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 93–110 (1950).

The author investigates the general theory of a zoom lens within the boundaries of Gaussian optics. Two lens systems image a fixed object plane on a fixed image plane. The question studied is, how do we have to change the position of the two lenses so that object and image plane remain conjugated, while the magnification varies?

M. Herzberger (Rochester, N. Y.).

Cotton, P., et Rouard, P. Mise au point bibliographique sur les propriétés optiques des lames minces solides. J. Phys. Radium (8) 11, 461–479 (1950).

Šifrin, K. S. **Scattering of light by large drops of water and polarization of light in rainbows.** Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 128–163 (1950). (Russian)

In questo studio viene affrontata la trattazione più esauriente, che possa aver luogo in base all'ottica geometrica, della diffusione della luce da parte di gocce d'acqua sferiche. Anche quando il diametro delle gocce è grande rispetto alla lunghezza d'onda, si rende conto così soltanto di metà della luce diffusa. Ma l'altra metà (diffratta) viene tutta concentrata in un piccolo angolo solido in avanti. L'autore divide la luce diffusa in vari ordini successivi; l'ordine k è quello dovuto ai raggi che hanno incontrato k volte la superficie della goccia. Se φ e ψ sono gli angoli d'incidenza e di rifrazione di un raggio, il suo angolo di diffusione nell'ordine k sarà $\beta^{(k)} = (k-2)\pi + 2[\varphi - (k-1)\psi]$. Facendo uso dei coefficienti di Fresnel per la riflessione e la rifrazione, vengono calcolate in funzione di $\beta^{(k)}$ le intensità $I_p{}^{(k)}$ e $I_s{}^{(k)}$ diffuse, corrispondenti ai due possibili stati di polarizzazione. Viene così costruita un'ampia tavola numerica, dove $\beta^{(k)}$ varia di grado in grado da 0 a 180° e k varia da 1 a 10. L'intensità dell'ordine k diviene infinita quando è soddisfatta la relazione $1 - [(k-1)/n] \cos \varphi / \cos \psi = 0$ e il corrispondente angolo $\beta^{(k)}$ è quello dell'arcobaleno di ordine k. Il flusso totale, naturalmente, rimane finito. In un'altra tavola numerica viene data in funzione di β l'intensità complessiva, dovuta a tutti gli ordini, e il suo grado di polarizzazione.

G. Toraldo di Francia (Firenze).

Rinner, Karl. **Abbildungsgesetz und Orientierungsaufgaben in der Zweimedienphotogrammetrie.** Österreich. Z. Vermessgswes. Sonderheft **5**, ii+46 pp. (1948).

The problem investigated by the author is the evaluation of distances of objects under water, photographed from the air. Three problems are especially investigated. The laws of image formation are developed and approximation formulas introduced based upon the fact that the underwater layer to be investigated has mostly a thickness of less than 1/100 of the object distance. With these approximation formulas the author calculated the important photogrammetric data. He proposes compensation systems for an optical compensation of the change of perspective and gives a theoretical solution under the assumption that the positions of three points are known. The second part of the paper contains the solution of the problem of stereoscopic photogrammetry. Here the knowledge of eight under water points is necessary to construct a model and the author suggests an approximation method for its solution.

M. Herzberger (Rochester, N. Y.).

Glaser, Walter. **Berechnung der optischen Konstanten starker magnetischer Elektronenlinsen.** Ann. Physik (6) **7**, 213–227 (1950).

L'equazione differenziale della traiettoria di un raggio in una lente magnetica può scriversi, introducendo opportune grandezze ridotte, nella forma adimensionale

$$(1) \qquad d^2r/dx^2 + k^2\beta(x)r = 0,$$

essendo k^2 un parametro che caratterizza la potenza della lente e dipende dall'intensità del campo, dalla sua estensione e dalla tensione acceleratrice. La traiettoria del raggio che incide parallelamente all'asse all'altezza 1 e che serve per determinare il fuoco e la distanza focale, può rappresentarsi con la serie $r = 1 - k^2 r_1(x) + k^4 r_2(x) - \cdots$; sostituendo nella (1) si trova una formula ricorrente per il calcolo dei successivi termini. Il primo termine dà l'approssimazione classica

per le lenti di debole potenza. Per le forti potenze la serie converge troppo lentamente e l'autore ricorre all'artificio di sostituire il campo magnetico effettivo con un campo fittizio, che gli si adatta fino al quarto ordine, con $\beta(x)$ della forma $1/(1+x^2)^2$. Con questa posizione la (1) si sa integrare in termini finiti. Il discostamento da tale forma di $\beta(x)$ può essere studiato con il classico metodo delle perturbazioni o con uno dei metodi variazionali noti per il calcolo degli autovalori della (1). *G. Toraldo di Francia* (Firenze).

Copson, E. T. **Diffraction by a plane screen.** Proc. Roy. Soc. London. Ser. A. **202**, 277–284 (1950).

The author now answers the objections raised by Bouwkamp [Dissertation, Groningen, 1941; these Rev. **8**, 179]. In the solution of the integro-differential equations, an arbitrary function arises and the author shows how the evaluation of this function is dependent on the order of the singularity of the wave functions at the edge of the screen. The three-dimensional problem of diffraction of a plane wave by a semi-infinite half plane is used to illustrate this remark. *A. E. Heins* (Pittsburgh, Pa.).

Braunbek, Werner. **Neue Näherungsmethode für die Beugung am ebenen Schirm.** Z. Physik **127**, 381–390 (1950).

The author discusses the problem of diffraction of a scalar plane wave by a plane screen or obstacle in the limit of large wave-lengths. *A. E. Heins* (Pittsburgh, Pa.).

Braunbek, Werner. **Zur Beugung an der Kreisscheibe.** Z. Physik **127**, 405–415 (1950).

The author applies his discussion in the paper reviewed above to a circular obstacle. *A. E. Heins*.

Toraldo di Francia, Giuliano. **Sezioni d'urto di schermi metallici.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 369–373 (1950).

The result of this paper is that the collision cross-section of a perfectly conducting plane screen for waves of given polarisation is equal to twice the transmission cross-section of the aperture in the complementary screen for waves polarised at right angles to the given waves. The proof involves the integral equation for the diffraction of electromagnetic waves by a plane screen and the rigorous formulation of Babinet's principle. *E. T. Copson* (Dundee).

Jones, D. S., and Pidduck, F. B. **Diffraction by a metal wedge at large angles.** Quart. J. Math., Oxford Ser. (2) **1**, 229–237 (1950).

This paper studies the effect of a line source radiation diffracted by a metal wedge. Two cases are considered, the parallel and perpendicular polarizations. The distinction between perfect conductor and metal is made by the authors. That is, a metal is considered by them to be a homogeneous conducting dielectric with complex refractive index. Boundary conditions similar to those for perfect conductors now hold. *A. E. Heins* (Pittsburgh, Pa.).

Svešnikov, A. G. **The principle of radiation.** Doklady Akad. Nauk SSSR (N.S.) **73**, 917–920 (1950). (Russian)

Various conditions are discussed which serve to make unique the solution of the wave equation for an infinite region, the condition of finiteness being insufficient when k is real. Ignatowsky [Ann. Physik (4) **18**(323), 495–522, 1078 (1905)] proposed, in a particular problem, to make k tend to the real value through complex values; this the author

terms the "principle of limiting absorption." Secondly, there is Sommerfeld's "radiation condition" [Jber. Deutsch. Math. Verein **21**, 309–353 (1912)]. Thirdly, Tihonov anp Samarskiĭ [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **18**, 243–248 (1948)] have suggested a "principle of limiting amplitude," recommending the limit as $t \to \infty$ of the amplitude of a certain solution of the time-dependent wave equation.

For wave propagation in an infinite slab $(0 \leq s \leq l)$, with $\Delta v + k^2 v = 0$ in the slab, v vanishing on the upper and lower faces of the slab, the author derives the following modified radiation conditions: $\rho^{\frac{1}{2}} v_m$ finite as $\rho \to \infty$, $\rho^{\frac{1}{2}}(\partial v_m/\partial \rho - i k_m v_m) \to 0$ as $\rho \to \infty$, where the v_m, k_m are given by $v = \sum_{m=1}^{\infty} v_m \sin m\pi z/l$, $k_m = k[1 - (\pi/lk)^2 m^2]^{\frac{1}{2}}$. Passing to general diffraction problems, the author gives integral representations by means of which, he states, the "principle of limiting absorption" may be justified. Similar considerations are stated to hold for Maxwell's equations.

F. V. Atkinson (Ibadan).

✳**Weber, Ernst.** **Electromagnetic Fields.** **Theory and Applications.** **Volume I.** **Mapping of Fields.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. xvi+590 pp. $10.00.

As the author states in the preface, it is hardly possible to add basically new material to the subject of electromagnetic theory. Yet, the utilization of electromagnetic phenomena has by no means reached the point of saturation. Hence a book giving a rather comprehensive survey of the methods and results of analysis would prove of value to students and teachers in advanced courses as well as to physicists and engineers in research and development. The subject matter of such a book might be naturally divided into two fundamental branches: one dealing with static electric and magnetic fields (potential theory); the other dealing with the dynamical interaction of electric and magnetic fields (theory of the wave equation). This division has been clearly recognized and followed by the author. The first volume, now being reviewed, presents a survey of the methods of mapping static electric and (stationary) magnetic field distributions as derived from potential functions. Analytical, experimental, graphical, and numerical methods are treated thoroughly. Of course, not all that is known on the subject can be included in one volume of moderate size. Yet, in the reviewer's opinion, the author has fairly well succeeded in his difficult task. There are more than three hundred problems for the student to solve, distributed over the various chapters. The book is an excellent reference work. [Unfortunately there is no author index.]

Chapters I and II contain a clear survey of the fundamental theory of static electric and magnetic fields. It is assumed that the reader possesses a general knowledge of electromagnetic theory as normally gained in an undergraduate course, and that he is familiar with vector notation. Rationalized Giorgi units are employed throughout the book. Chapter III is devoted to general field analogies: stationary electric current, stationary temperature, fluid dynamic, and gravitational fields. Fields of simple geometries are treated in chapter IV: systems of point and quasi point charges, line and quasi line charges, line and quasi line currents, distributed charges and currents. Experimental mapping methods are treated in chapter V, including the electrolytic trough and the rubber membrane. Field plotting methods are discussed in chapter VI: graphical methods, electrical and magnetical images, inversion, relaxation method for

two-dimensional fields. The last two chapters, covering about half the text proper, deal with analytic solutions of potential problems in two and three dimensions. A wealth of material is presented here, which makes these chapters most appealing to the mathematical physicist. Chapter VII includes the theories of conjugate potential functions, conformal mapping, general boundary-value problems. Chapter VIII surveys the various possible orthogonal coordinate systems for which the potential equation is separable, with applications of Bessel, Weber, Mathieu, Legendre, and other functions of mathematical physics. There are six appendices providing additional information as to (1) letter symbols for electrical quantities, (2) conversion tables for units, (3) fundamentals of vector analysis, (4) general bibliography, (5) Bessel functions, (6) Legendre functions [in the familiar notation $P_n{}^m$ of Legendre functions, the author calls n the order and m the degree, which is different from established usage]. The book is well written, excellently printed, and equipped with many illustrations.

C. J. Bouwkamp (Eindhoven).

✳**White, F. W. G.** **Electromagnetic Waves.** 4th ed. Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, N. Y., 1950. viii+108 pp. $1.25.

This edition has been revised to include a brief account of the work of C. G. Darwin [Proc. Roy. Soc. London. Ser. A. **146**, 17–46 (1934)].

Keller, Joseph B., and **Preiser, Stanley.** **Determination of reflected and transmitted fields by geometrical optics. II.** New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-20**, i+i+10 pp. (1950).

This paper extends earlier work of J. B. Keller and H. B. Keller [same report series, Rep. No. **EM-13** (1949) = J. Opt. Soc. Amer. **40**, 48–52 (1950); these Rev. **11**, 561]. The authors compute the amplitudes of the reflected and transmitted fields for an arbitrary incident wave front impinging on an arbitrary interface of two media.

C. J. Bouwkamp (Eindhoven).

Keller, Herbert B., and **Keller, Joseph B.** **A point dipole in spherically symmetric media.** New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-16**, i+21 pp. (1950).

The authors' abstract reads as follows: "The problem of determining the electromagnetic field due to a radiating point dipole at the center of a spherically symmetric medium is formulated in terms of the vector potential. When the medium is piecewise constant the exact solution is obtained by means of a recursion formula. For the continuously variable medium the problem is reduced to the solution of a Riccati equation. Some special configurations and media are considered and expressions for the reflection and transmission coefficients are obtained." [Unfortunately the paper in its present generality is not correct. In a letter to the reviewer the authors indicate as to what extent their results are considered of value. They intend to publish a supplementary report on the subject, so that a complete review of the paper seems premature.]   *C. J. Bouwkamp.*

Kahan, T., and **Eckart, G.** **On the existence of a surface wave in dipole radiation over a plane earth.** Proc. I.R.E. **38**, 807–812 (1950).

This is the fifth of a series of papers on Sommerfeld's surface wave by the same authors [C. R. Acad. Sci. Paris

226, 1513–1515 (1948); **227**, 969–970 (1948); J. Phys. Radium (8) **10**, 165–176 (1949); Physical Rev. (2) **76**, 406–410 (1949); these Rev. **9**, 637; **11**, 143]. Though there is a slight improvement in the manner of presentation, the authors' ultimate views on the surface wave should, in the reviewer's opinion, be rejected [cf. the reviewer's criticisms in the cited reviews]. Though they cite a paper by F. Rellich [Jber. Deutsch. Math. Verein **53**, 57–65 (1943); these Rev. **8**, 204] they do not appear to have assimilated it.

C. J. Bouwkamp (Eindhoven).

Fok, V. A. Theory of the propagation of radio waves from an elevated source in an unhomogeneous atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **14**, 71–94 (1950). (Russian)

L'autore generalizza gli studi già eseguiti per una sorgente a terra e atmosfera non omogenea [gli stessi Izvestiya. Ser. Fiz. **12**, 81–97 (1948); questi Rev. **10**, 89] e per una sorgente elevata e atmosfera omogenea [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **19**, 916–929 (1949); questi Rev. **11**, 563], passando al caso combinato, enunciato nel titolo. Lo spazio non consente di esporre i dettagli dei calcoli, che seguono una linea analoga a quella dei lavori citati. Il vettore di Hertz viene ora posto nella forma

$$U = e^{i(1+a)ks}[sa \sin (s/a)]^{-\frac{1}{2}}V(x, y_1, y_2, q)$$

dove i simboli hanno un significato già noto [questi Rev. **11**, 563; per errore di stampa l'esponente numerico nella formula è ivi $\frac{1}{3}$, anziché $-\frac{1}{2}$], salvo la nuova costante α, dipendente dalla distribuzione della costante dielettrica nell'atmosfera. Lo studio della funzione V viene ricondotto a quello di un'equazione differenziale del tipo

$$(d^2f/dy^2) + [y + r(y) - t]f = 0$$

la cui soluzione generale viene data sotto forma d'integrale o di serie. Viene fatta un'applicazione alla superrifrazione, in cui l'atmosfera si comporta come una guida d'onda. L'autore segnala l'analogia fra il problema studiato e quello dello sparpagliamento di un pacchetto d'onde nella meccanica quantistica. Gli stati stazionari corrispondono ai modi che si propagano lungo la guida d'onda.

G. Toraldo di Francia (Firenze).

Kelleher, K. S. Relations concerning wave fronts and reflectors. J. Appl. Phys. **21**, 573–576 (1950).

The problem presented and solved by the author is: "Given any two of the three surfaces, incident wave front, reflector and reflected wave front, find the third." By employing a concise notation of differential geometry, he treats several practical problems, including the reflection of a conical wave front by a parabolic cylinder and the construction of reflectors that yield virtual point sources or plane waves.   *C. J. Bouwkamp* (Eindhoven).

Knol, K. S. Propagation of electromagnetic waves in rectangular wave guides. Nederl. Tijdschr. Natuurkunde **16**, 41–58 (1950). (Dutch)

This is an expository paper read before the Dutch Physical Society.   *C. J. Bouwkamp* (Eindhoven).

Patry, J. Theoretische Untersuchungen über die Strahlungsimpedanz von Sendeantennen. Schweiz. Arch. Angew. Wiss. Tech. **16**, 138–147 (1950).

The author considers a compound antenna made up of a set of linear elements, each parallel to one axis of a rectangular coordinate system. Hallén's integral equation method is used to determine the interaction between parallel and perpendicular elements, and explicit formulas for the input resistance at resonance and at antiresonance are obtained. The effect of the gap at the center of each element is included by introducing a modified parameter, $\Omega' = \log \{4(L^2 - X_0^2)/\rho[2X_0 + (4X_0^2 + \rho^2)^{\frac{1}{2}}]\}$, where $2X_0$ is the gap width, in place of the usual $\Omega = 2 \log (2L/\rho)$. Some numerical results for various arrangements of the antenna elements are also tabulated.   *M. C. Gray.*

Jaeger, J. C., and Westfold, K. C. Transients in an ionized medium with applications to bursts of solar noise. Australian J. Sci. Research. Ser. A. **2**, 322–334 (1949).

"Exact solutions of a number of transient problems on linear propagation in a homogeneous ionized medium without magnetic field are given. The Fourier transforms of the solutions are discussed. The effect of inhomogeneity of the medium is studied, and it is shown that if a localized disturbance takes place in the solar atmosphere at a level at which the collision frequency is ν and the frequency of plasma oscillations is ω_0, radiation will be emitted on all frequencies greater than ω_0 with intensity determined by the nature of the disturbance and with damping determined by ν. There is a small difference between the times of arrival on different frequencies, and on each frequency there is a direct wave and an echo wave which arrives some seconds later. Many of the phenomena of bursts of solar noise are consistent with those predictions." From the author's summary.   *N. Levinson* (Cambridge, Mass.).

✴**Gavrilov, M. A. Teoriya releĭno-kontaktnyh shem. [Theory of Relay-Contact Schemes].** Izdat. Akad. Nauk SSSR. Moscow-Leningrad, 1950. 303 pp.

This is the first book to be published on the theory of switching circuits. The author has tried to present a theory which applies to as many as possible of the different kinds of switching circuits in common use. Simple combinational circuits, relay circuits with various kinds of sequential behavior, circuits using rectifiers, and circuits using special purpose switching elements such as selector switches, relays with multiple windings, marginal relays, relays with "make before break" contacts are all discussed with the aid of Boolean algebra.

The book starts with introductory chapters describing the different kinds of switching circuits. Then the algebra of logic is introduced and applied to the analysis and synthesis of switching networks. A large part of the book is devoted to the problem of transforming a switching network into an equivalent simpler network; these results necessarily are applicable only under fairly restrictive circumstances. The last chapter is devoted to the synthesis of some practically useful networks (decoding circuits, adders, etc.). Much of the material given is drawn directly from the author's own papers on switching. A large number of examples are worked out in the text and the book is well illustrated with 217 diagrams.   *E. N. Gilbert.*

Cotte, Maurice. Emploi d'une impulsion pour l'essai d'un système de transmission linéaire. C. R. Acad. Sci. Paris **231**, 117–119 (1950).

This is a second note on the subject [see the same C. R. **228**, 1693–1695 (1949)] giving an approximation to the expression of distortion of a signal by a linear transmission circuit in terms of the response to a very short pulse of unchanging sign.   *E. Weber* (Brooklyn, N. Y.).

Lemaître, G. Application des méthodes de la mécanique céleste au problème de Störmer. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 83–97 (1949).

The Hamiltonian for the nonplanar motion of a charged particle in a dipole magnetic field is developed in a trigonometric series analogous to those in celestial mechanics. Two different formulations are given, the first for perturbations about a harmonically oscillating orbit and the second about an orbit obtained when the distance to the dipole is constrained to be constant. To these Hamiltonians the method of Delaunay [cf. Tisserand, Traité de Mécanique Céleste, vol. 3, Gauthier-Villars, Paris, 1894, p. 181] is applied in general outline.   *R. G. Langebartel* (Saltsjöbaden).

Lemaître, G. Application des méthodes de la mécanique céleste au problème de Störmer. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 76–82 (1950).

Continuing the work in his earlier paper [see the preceding review] the author introduces a small modification in his first formulation and carries out the perturbational method to arrive at five algebraic equations defining the constants for a first order theory of periodic trajectories.

R. G. Langebartel (Saltsjöbaden).

Quantum Mechanics

Faure, Robert. Correspondance mécanique classique-mécanique ondulatoire. J. Math. Pures Appl. (9) **28**, 193–285 (1949).

A wave equation is written to correspond to a mechanical system whose (nonrelativistic) kinetic energy and potential energy are given functions of generalized coordinates and velocities; the case of a vector potential giving the effect of an electromagnetic field is included. The substitution $\psi = a e^{i\varphi/\hbar}$ leads to two equations for the functions a and φ. In the lowest order in $\hbar$, the geometrical-optical limit, the classical Hamilton-Jacobi equation is obtained. The paper is mainly concerned with studying, from this point of view, the relation between first integrals of the classical problem and constants of the motion for the quantum-mechanical problem.   *W. H. Furry* (Cambridge, Mass.).

Kou, T. T. Discussion on the behavior of an electron enclosed in a sphere. Chinese J. Phys. **7**, 241–248 (1949). (English. Chinese summary)

The author solves the Schrödinger equation for a single electron confined to the interior of a sphere of radius r_0. The potential energy is taken to be zero inside the sphere and infinite on the boundary. The Schrödinger equation is then the ordinary wave equation, solutions are ordinary spherical waves, and the radial factor is of the form $r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr)$, where k is selected so that the radial factor vanishes on the boundary $r = r_0$. The spherical Bessel function $r^{-\frac{1}{2}}J_{n+\frac{1}{2}}$ is an elementary function, and its zeros are readily found. The author finds various explicit formulas for the first few cases.   *O. Frink* (State College, Pa.).

Snyder, Hartland S. Quantum field theory. Physical Rev. (2) **79**, 520–525 (1950).

This paper is a sequel to an earlier paper [same Rev. (2) **78**, 98–103 (1950); these Rev. **11**, 632]. Generalizing the arguments of the earlier paper, the author proposes a method by which the divergences of quantized field theories may be completely eliminated. All virtual processes occurring in such theories may be eliminated by a succession of unitary transformations S, the effects of the virtual processes then appearing explicitly in the transformed Hamiltonian operator H'. In the usual treatment, H' contains a variety of divergent integrals which are interpreted as infinite mass and charge renormalizations. The author replaces each S by a modified unitary operator $S(\epsilon)$ depending on a real parameter ϵ, with the following properties: (i) for nonzero ϵ, the $H'(\epsilon)$ obtained by applying $S(\epsilon)$ are free from divergences; (ii) $S(\epsilon) \rightarrow S$ as $\epsilon \rightarrow 0$, the convergence being nonuniform over the various matrix elements of $S(\epsilon)$; (iii) as $\epsilon \rightarrow 0$, $H'(\epsilon)$ tends to a finite limit H'' which is divergence-free. The author's suggestion is to use H'' as the physically correct transformed Hamiltonian.

In the reviewer's opinion, the program could be worked out consistently, and would give results identical with those of the usual treatment, so far as finite observable effects are concerned. However, to prove that this is so, the author will need to show in detail (i) that the operators $S(\epsilon)$ and H'' can be constructed, and (ii) that the physically observable effects contained in H'' are independent of the arbitrary functions appearing in the definition of $S(\epsilon)$. These are difficult questions which are discussed only very incompletely in the paper under review.   *F. J. Dyson.*

Salam, Abdus. Differential identities in three-field renormalization problem. Physical Rev. (2) **79**, 910–911 (1950).

It is known that in spinor quantum electrodynamics all divergences can be absorbed into unobservable renormalizations of the mass and charge of the electron. The proof of this is greatly simplified by using certain differential identities due to J. C. Ward [same Rev. (2) **77**, 293 (1949); **78**, 182 (1950); these Rev. **11**, 632]. The author sketches a set of identities similar to Ward's, which will simplify the discussion of the renormalization problem for a system of three fields in interaction (nucleon field interacting simultaneously with a Maxwell field and with a charged zero-spin meson field). The three-field problem is very much more complicated than the case of spinor electrodynamics. A complete discussion of the three-field problem will be published later by the author.   *F. J. Dyson.*

Nishijima, K. Note on the elimination of the normal-dependent part from the Hamiltonian. Progress Theoret. Physics **5**, 331–332 (1950).

In the interaction representation, the Hamiltonian describing the interaction between two fields is customarily of the form $H(x) = -L(x) + K_{\mu\nu}(x)N_\mu N_\nu$, where $L(x)$ is the interaction term in the Lagrangian and is a scalar, while N_μ is a unit vector normal to the reference surface. It is found that in applications of the formalism, the effects of the term in $N_\mu N_\nu$ are always cancelled exactly by second-order effects from the term $-L(x)$ [see P. T. Matthews, Physical Rev. (2) **76**, 684–685 (1949)]. The author gives a simple and general explanation of this cancellation.

F. J. Dyson (Birmingham).

Rumer, Yu. B. Physical content of 5-optics. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 199–205 (1950). (Russian)

This paper is a summary, written on a less technical level, of the results of two previous papers of the author [same Žurnal **19**, 86–94, 207–214 (1949); these Rev. **10**, 580].

It is explicitly addressed to experimental as well as to theoretical physicists. The consequences of the theory of 5-optics are stated to include (i) the existence of a magnetic field in massive neutral rotating bodies, (ii) the existence of particles with twice the mass and twice the charge of an electron, (iii) significant departures from the predictions of orthodox quantum electrodynamics in high-energy radiation phenomena. Unfortunately none of these consequences is deduced in a quantitative form so as to be susceptible to direct experimental test.   *F. J. Dyson.*

Pais, A., and Uhlenbeck, G. E.   On field theories with non-localized action.   Physical Rev. (2) **79**, 145–165 (1950).

The effects of generalizing the equations of field theory to higher order differential or to integral equations are investigated. Equations of the form $F(\Box)\psi=\rho$ are considered. When F is the form $F(\Box)=\prod_i(\Box-\kappa_i^2)$ (with the product finite or infinite), it is shown that even though classical divergences may be eliminated there still remain serious quantum mechanical difficulties. Self energies may be made finite but only at the expense of negative energy troubles. Moreover, this procedure, when applied to the Dirac equation, does not affect the charge renormalization. On generalizing F further to integral functions it is indeed found that finiteness may be achieved, but at the cost of giving up the notion of strict causality. In dealing with these questions rather elegant methods are introduced to treat the quantum mechanical problems involving higher order differential equations.   *K. M. Case* (Ann Arbor, Mich.).

Thirring, W.   On a fourth-order meson-equation.   Philos. Mag. (7) **41** 653–662 (1950).

An iterated Klein-Gordon operator is used to exemplify a field theory involving higher order derivatives. This is a special case of the class of theories considered by Pais and Uhlenbeck [see the preceding review]. The negative energy difficulties found by these authors have not been touched here. A detailed calculation of the self energy of a nucleon is made using the covariant techniques. While the higher derivatives result in a small, finite second order energy, the higher order corrections still diverge.   *K. M. Case.*

Thirring, Walter.   Symmetrische Quantisierung.   Acta Physica Austriaca **4**, 125–128 (1950).

The commutator of two field operators is assumed to be a *c*-number function of the space time coordinates of the fields in the case of no interaction. Using the postulate of invariance under the inhomogeneous Lorentz group, the antisymmetry of the commutator, the fact that the commutator must also satisfy the field equations, and a tacit finiteness requirement, the author then derives this *c*-number function to within a constant. This is only done for the case of a scalar field, but as is pointed out, the same procedure will work for the commutators of more general fields.
                                      K. M. Case (Ann Arbor, Mich.).

Thirring, W.   Quantization of higher order equations.   Physical Rev. (2) **79**, 703–705 (1950).

The author claims to simplify the problem of quantization of higher order equations. The condition that the interaction and Heisenberg representations must be equivalent leads directly to the determination of the commutator function. The method is illustrated by a scalar field $\phi(x)$ satisfying the vacuum equation $\prod_{i=1}^n(\Box^2-m_i^2)\phi(x)=0$. This leads in a natural way to the regularization conditions of Pauli and

Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301], and in the case $n=2$, $m_1=m_2$, to a non-singular meson field proposed recently by Bhabha [same Rev. (2) **77**, 665–668 (1950); these Rev. **11**, 764]. The paper ends with a brief indication of how the method may be applied to spinor fields.   *A. J. Coleman.*

Petiau, Gérard.   Sur l'unification des représentations dans la théorie du corpuscule de spin total maximum $2\frac{h}{2\pi}$.

Cahiers de Physique no. **33**, 1–16 (1948).

A particle of spin $2h$ where h is Planck's constant divided by 2π is represented by a wave function which transforms as a four-index four-component spinor. The author discusses the tensor equations equivalent to the wave equation satisfied by this spinor. Since the five-dimensional rotation group may be represented by four-component spinors it is evident that there is a correspondence between five-dimensional tensors and linear forms of four-index spinors. The author uses this correspondence in discussing the interaction of the field of the particle of spin $2h$ with other fields and shows that if the wave equation with interaction is to be invariant under rotations in five dimensions the number of interaction constants is restricted.   *A. H. Taub* (Urbana, Ill.).

Yang, C. N., and Tiomno, J.   Reflection properties of spin $\frac{1}{2}$ fields and a universal Fermi-type interaction.   Physical Rev. (2) **79**, 495–498 (1950).

It is pointed out that four types of four-component spinors exist. Each type is distinguished from another by the nature of the spin image of the improper transformation $x^{i\prime}=-x^i$ ($i=1, 2, 3$), $t^\prime=t$. The consequences of assuming that different particles are represented by spinor fields of different types are discussed with particular reference to the scalars representing the interaction between the particles.
                                      A. H. Taub (Urbana, Ill.).

Visconti, Antoine.   Application de la transformation de Laplace à l'équation de l'opérateur d'évolution.   C. R. Acad. Sci. Paris **231**, 333–335 (1950).

Using Feynman's representation K for the evolutionary operator $\mathfrak{U}$, the author obtains an explicit expression of K by a Laplace transform.   *C. C. Torrance.*

Visconti, Antoine.   Remarques sur une solution de l'équation d'ondes.   C. R. Acad. Sci. Paris **231**, 507–509 (1950).

It is shown that a particular solution of the wave equation (connected with the evolutionary operator $\mathfrak{U}$) can be written directly by means of de Broglie waves on which a certain diffusion law is imposed.   *C. C. Torrance.*

Wildermuth, Karl.   Der analytische Zusammenhang zwischen den Streumatrix-Elementen und den diskreten stationären Zuständen in der Heisenbergschen S-Matrix-Theorie.   Z. Physik **127**, 85–91 (1950).

S. T. Ma [Physical Rev. (2) **69**, 668 (1946)] has given an example in which the Heisenberg-Kramers method of finding stationary states, by seeking the zeros of the analytical extension of the eigenvalues $S'(E)$ of the S-matrix as functions of E, gives rise to redundant eigenvalues in addition to the actual ones. The present author gives a new necessary condition on the sign of $S'(E)$ which eliminates the spurious eigenvalues in Ma's example. The application of this criterion requires a knowledge of $S'(E)$ in an infinitesimal neigh-

bourhood of the positive E-axis. It is not claimed that this new condition is sufficient to guarantee true eigenvalues.

A. J. Coleman (Toronto, Ont.).

Wildermuth, Karl. Das analytische Verhalten der asymptotischen Wellenfunktion und die S- bzw. η-Matrix für mehrere Teilchen. I. Z. Physik **127**, 92–121 (1950).

A study is made of the analytical behaviour of the elements of Heisenberg's S-matrix for two one-dimensional particles. One is free, the other bound to the origin by a delta type potential, and they interact with a delta type potential. A relation (29e) is obtained giving the matrix elements corresponding to pure scattering as a contour integral of matrix elements describing ionisation or dissociation of the second particle. A. J. Coleman.

Wildermuth, Karl. Das analytische Verhalten der asymptotischen Wellenfunktion und die zugehörige S-Matrix für mehrere Teilchen. II. Z. Physik **127**, 122–152 (1950).

The study described in the preceding review is extended to the case of two three-dimensional particles. The validity of the relation (29e) is verified here and the validity of its generalization for any n-particle problem is conjectured. It is shown that the two examples lend credence to the following conjecture: "Those elements of the S-matrix which describe pure scattering are analytic functions of the energy E whose only singularities are logarithmic, occurring at those values of E at which a particle can be ionized or new particles created. The matrix elements representing creation or annihilation of particles cannot depend analytically on E." Throughout the two papers consistent use is made of momentum space for all calculations. This, not only in order to simplify the treatment, but also because of Heisenberg's argument that the existence of a minimum length of the order 10^{-13} cm., would imply that it is nonsensical to attempt to represent nuclear forces, which are effective only at such ranges, in coordinate space. A. J. Coleman.

Wildermuth, K. Die Konstruktion von relativistisch invarianten η- bzw. S-Matrizen für mehrere Teilchen. Z. Physik **127**, 551–562 (1950).

Using results from the two papers reviewed above, the author studies a three-particle problem. The peculiar interest of this paper lies in the fact that he succeeds in giving matrix elements for η ($S = e^{i\eta}$) which are Lorentz invariant. Particular matrix elements corresponding to pure scattering, to the combining of two particles, etc., are studied in detail. A comparison of the η and Hamiltonian formalisms leads the author to predict that a thoroughly satisfactory theory will necessarily involve many-particle-interactions.

A. J. Coleman (Toronto, Ont.).

Falkoff, David L., and Uhlenbeck, G. E. On the directional correlation of successive nuclear radiations. Physical Rev. (2) **79**, 323–333 (1950).

The correlation in angle of the radiation emitted in two successive nuclear transitions is considered. It is shown that two types of information are needed for the theoretical calculation of a correlation: (a) information regarding the rotation properties of the nuclear states and of the emitted radiation, (b) the specific interaction Hamiltonian. An attempt is made to carry the calculation of the correlation function as far as possible using information of type (a). Here it is assumed that the radiations are each emitted in states of a single angular momentum. Putting the resulting

formulas into a canonical form involving constants to be determined by (b) makes the calculation of specific correlation such as γ-γ, β-γ, α-γ transitions particularly simple and avoids needless repetition of computations. The sums involved are given for emitted particles in states of angular momentum 1 or 2. The assumption which was made here that there is no interference between transitions via different intermediate levels has since been proven by Spiers [same Rev. (2) **80**, 491 (1950)]. K. M. Case.

Falkoff, David L., and Uhlenbeck, G. E. On the beta-gamma-angular correlation. Physical Rev. (2) **79**, 334–340 (1950).

The formulae found in the paper reviewed above are specialized to the case of successive β-γ transitions. It is expected that these β-γ correlations will be of use in distinguishing between the different β-decay interactions which have been proposed. Moreover, it is useful in discovering the angular momentum of nuclear energy levels. Using the Konopinski and Uhlenbeck [same Rev. (2) **60**, 308–320 (1941)] classification of transitions, the following general results are obtained. 1. For allowed β-transitions followed by any other radiation, there can be no angular correlation. 2. For any forbidden β-transition having an allowed spectrum shape followed by any other radiation there can be no angular correlation. 3. For β-particles near the low energy end of the β-spectrum there is no β-γ angular correlation; β particles having energy near the maximum will yield the strongest correlation. Specific formulas for the correlation function for various interactions and forbiddenness are found. K. M. Case (Ann Arbor, Mich.).

Vyatskin, A. Ya. On the rôle of the surface and volume effect in the secondary electron emission of metals. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 547–556 (1950). (Russian)

The potential energy of an electron inside the metal is taken to be a three-dimensional periodic function; that for an electron outside is constant. No special variation of potential at the surface is assumed. The effect of inelastic scattering is supposedly approximated by taking the metal to be bounded also by another surface parallel to the actual surface. The primary electron is treated as a free particle, and the interaction is treated in Born approximation. The energy-distribution found for the secondary emission shows small periodic increases (volume effect) superposed on the general course of the curve (surface effect). Experimental results show rough agreement with these predictions.

W. H. Furry (Cambridge, Mass.).

Vyatskin, A. Ya. Inelastic scattering of electrons passing through metals. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 557–565 (1950). (Russian)

The inelastic scattering of an electron traversing a metal, caused by Coulomb interaction with electrons in the metal, is calculated by the Dirac method of variation of parameters. The calculation shows considerable formal analogy with the Weisskopf-Wigner calculation of line-breadth; the probability amplitude for the electron's not yet having suffered inelastic scattering is obtained as $e^{-\gamma t}$, and the mean free time against inelastic scattering is then $1/2\gamma$. Transitions of conduction electrons between zones are found to be much more effective than transitions within a zone. The mean free path against inelastic scattering does not depend strongly on the electron's energy, and the total path is

roughly proportional to the energy; on the other hand, these quantities vary inversely as the fourth power of the lattice spacing.   *W. H. Furry* (Cambridge, Mass.).

Thermodynamics, Statistical Mechanics

Popoff, Kyrille. Observations de nature mathématique concernant le second principe de la thermodynamique. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 39, 217–220 (1943). (Bulgarian. French summary)

Kohler, Max. Entropiesatz im inhomogenen verdünnten Gas. Z. Physik 127, 201–208 (1950).

The author obtains an expression for the entropy-density increase in a rarefied gas arising from the second approximation stress and heat flux terms. In order to derive it he uses the results of Burnett and Chapman-Cowling [The Mathematical Theory of Non-uniform Gases, Cambridge University Press, 1939; these Rev. 1, 187] concerning the form of these terms. Thus he obtains a result, equation (19), which differs markedly from the classical entropy equation of thermodynamics, in that in addition to the total stress and heat flux terms there are two others of precisely the same form but containing the second approximation stress and heat flux instead. [The reviewer has not repeated the author's calculations, but he regards this result as one which should be carefully checked before it can be accepted.] By using his entropy equation and some formulae of Burnett for the first approximation to the distribution function, the author obtains numerical values for the coefficients of the linear terms in the second approximation for the stress and heat flux, both for the case of Maxwellian molecules and for the case of rigid spherical molecules.   *C. Truesdell.*

Kohler, Max. Eine Symmetriebeziehung in der Theorie der inhomogenen verdünnten Gase. Z. Physik 127, 215–220 (1950).

The author shows that in the second order stress and heat flux components derived from the kinetic theory of perfect monatomic gases by Enskog's method of integration of the Boltzmann equation one particular numerical coefficient in the stress formula is always equal to another particular numerical coefficient in the heat flux formula, whatever the molecular force law may be. In the notation of Chapman and Cowling [The Mathematical Theory of Non-uniform Gases, Cambridge University Press, 1939, § 15.3; these Rev. 1, 187], the result is $\bar{\omega}_3 = \theta_4$.   *C. Truesdell.*

Bass, Jean. Lois de probabilité, équations hydrodynamiques et mécanique quantique. Revue Sci. 86, 643–652 (1948).

Starting with the pseudo-kernel $S(x, x', t)$, $(x) = (x_1, \cdots, x_N)$, $(x') = (x_1', \cdots, x_N')$, of the well-known quantum statistical operator, which determines the statistics of a system of N degrees of freedom in a given state (pure or not), the author introduces in a natural manner Wigner's pseudo-density $f(x, u, t)$, $(u) = (u_1, \cdots, u_N)$, which is equivalent with $S(x, x', t)$ through appropriate Fourier transformations. He shows, with J. Yvon, that the mean $\bar{C}$ of any observable C can be expressed by writing $\bar{C}(t) = \int\int c(x, u, t) f(x, u, t) dx\, du$, with an appropriate function $c(x, u, t)$ determined from C. This expression of $\bar{C}$ has the same form as in classical probability, but the pseudo-density $f(x, u, t)$ can be negative as

well as positive. From the equation of $S(x, x', t)$ in time t (essentially Schroedinger's equation), an equation of evolution of $f(x, u, t)$ is derived; it is formally similar to Liouville's equation of continuity as it is generalized to stochastic hydrodynamics, and becomes identical with it when terms in positive powers of Planck's constant h are dropped. From this point, generalizations of stochastic hydrodynamical equations or their formal analogues are derived under broad natural assumptions. Born and Green's equation is obtained as a special case.   *B. O. Koopman.*

✱**Bass, J.** Applications de la mécanique aléatoire à l'hydrodynamique et à la mécanique quantique. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 227, vii+143 pp. (1949).

A clear and detailed study and critique of the relation between the stochastic particle and (1) classical hydrodynamics, (2) nonrelativistic particle quantum mechanics, (3) quantum hydrodynamics. The author brings the subject up to date on the basis of his own investigations, incorporating those of E. Arnous, J. Yvon, G. Dedebant, P. Wehrle, E. Wigner, M. Born, H. S. Green, and others, full references being given.

The stochastic particle considered is a particle of unit mass in n-space whose coordinates $(X) = (X_1, \cdots, X_n)$ and velocity $(U) = (U_1, \cdots, U_n)$ form a $2n$-vector stochastic process in the time parameter t ($X_i = X_i(t)$, $U_i = U_i(t)$) and where $U_i = dX_i/dt$ and is continuous in t (all in the sense of convergence in mean square over the $2n$-space, weighted by the probability distribution function $F(x, u, t)$ in that space). When $n = 3$ and when the probability density $f(x, u, t)$ exists ($f(x, u, t)dx\, du = d_x d_u F(x, u, t)$), the conceptual basis of relation (1) runs as follows. Firstly, given a stochastic particle, if one defines the density $\rho(x, t) = \int f(x, u, t) du$ and the velocity $v_i(x, t) = \int u_i f(x, u, t) du$, one has four of the fundamental quantities of hydrodynamics. Conversely, given a turbulent fluid, the density $\rho(x, t)$ and the statistical density of velocities at a point lead to $f(x, u, t)$, but the author emphasizes that this part of the relationship (1) has not at the present time been worked out in sufficient quantitative detail and he applies himself instead to the study of the stochastic particle-into-fluid relation. Here it is necessary to define the forces (pressure) and thermodynamic quantities, and to derive the 4 equations of hydrodynamics and that of thermodynamics, i.e., to establish the equations of transfer (transfer of mass, momentum, and mechanical-thermodynamic energy). Everything is made to hinge on the analysis of the rate of change of the mean $\bar{G}$ of the stochastic variable $G(t) = g(X(t), U(t))$, where $g(x, u)$ is a bounded measurable complex-valued function on the $2n$-space of (x, u) (i.e., $g \epsilon L_\infty$). From the earlier hypotheses regarding (X, U), and on the assumption that $g(x, u)$ has continuous first and second partial derivatives, it is shown that $d\bar{G}/dt - \sum_{i=1}^n \overline{U_i g_i(X, U)}$ exists ($g_i(x, u) = \partial g(x, u)/\partial x_i$; bars denote means over (x, u)-space, weighted by $F(x, u, t)$). Henceforth the author imposes the additional hypothesis that this expression is equal to $\overline{AG}$, where A is an operator (which may contain t) defined over L_∞ (or at least over subsets of sufficiently differentiable functions in L_∞). Accordingly, all the equations of transfer are derived from $d\bar{G}/dt - \sum_i \overline{U_i g_i(X, U)} = \overline{AG}$, with various choices of G, i.e., of $g(x, u)$. The last general hypotheses are the following: (i) A is linear over L_∞; (ii) if $g(x, u) = 1$, then $Lg = 0$; (iii) if $g(x, u) = g_1(x)g_2(u)$, then $Ag = g_1(x)Ag_2(u)$ for all $g_1(x), g_2(u) \epsilon L_\infty$ (and, presumably, sufficiently differentiable); (iv) A has an adjoint A' in L_1.

When the stochastic particle is such that a probability density $f(x, u, t)$ exists and the first two moments of U are finite, all the equations of classical hydrodynamics (together with thermodynamic relations) are obtained, pressure, temperature, and internal energy being appropriately defined. Many other special cases are considered.

Turning to (2), the author shows that, whereas a quantum particle $(\mathbf{x}, \mathbf{p})$ cannot be regarded as a stochastic particle (X, U) in the earlier sense (operators $\mathbf{x}$, $\mathbf{p}$ not commuting), still, by introducing Wigner's pseudo-density of probability (sometimes negative) the formal probability relations in stochastic processes are valid. In (3) he derives the quantum hydrodynamical equations (including those of Born and Green) by applying the considerations in (1) formally to the pseudo-density formulas, the operator A now being derived from Schrödinger's equation. [For more details concerning (2) and (3) see the preceding review.]

B. O. Koopman (New York, N. Y.).

Born, M. The foundation of quantum statistics. Nuovo Cimento. (9) 6, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 163–170 (1949).

✴**Born, M., and Green, H. S. A General Kinetic Theory of Liquids.** Cambridge, at the University Press, 1949. vii+98 pp. $2.25.

A reproduction of six papers by the authors on the subject of the title [Proc. Roy. Soc. London. Ser. A. 188, 10–18 (1946); 189, 103–117 (1947); 190, 455–474 (1947); 191, 168–181 (1947); 192, 166–180 (1948); these Rev. 9, 402, 401; ibid. 194, 244–258 (1948)]. An appendix contains notes, corrections and additions.

Born, Max. Le seconde principe de la thermodynamique déduit de la théorie des quanta. Ann. Inst. H. Poincaré 11, 1–13 (1949).

A review (without detailed proofs) is given of an attempt to establish the second law of thermodynamics within the framework of the theory of liquids due to the author and Green [see the preceding review]. The classical theory is first put in a form very analogous to the quantum theory. It is pointed out, however, that the second law may be ascribed to different sources in classical and quantum statistical mechanics. In the former case the law of increasing entropy comes from our lack of knowledge of details of the motion of our system (details which are in principle available), while in the latter case the law of increasing entropy arises from the fundamentally statistical nature of the laws of quantum mechanics. With this interpretation, the quantum mechanical derivation of the second law becomes very simple. Using the standard probability definition of entropy and the well-known quantum mechanical law for the rate of change of probability of a slightly perturbed system, it is shown directly that the entropy always increases.

J. M. Luttinger (Madison, Wis.).

Irving, J. H., and Kirkwood, John G. The statistical mechanical theory of transport processes. IV. The equations of hydrodynamics. J. Chem. Phys. 18, 817–829 (1950).

The phenomenological equations of hydrodynamics, the continuity equation, equation of motion, and equation of energy transport are derived from the principles of classical statistical mechanics. The starting point is Liouville's equation for the density of points in phase space. Expressions are obtained for the stress tensor and heat current density

in terms of the intermolecular forces and pair distribution function of molecules of the system of interest. The pair distribution function must be derived through a separate statistical mechanical calculation. Although the results are quite similar to those of Born and Green [Proc. Roy. Soc. London. Ser. A. 190, 455–474 (1947); these Rev. 9, 402] they are derived and expressed in terms of Kirkwood's time smoothing process [same J. 14, 180–201 (1946)].

E. W. Montroll (College Park, Md.).

Kirkwood, J. G. The statistical mechanical theory of irreversible processes. Nuovo Cimento (9) 6, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 233–239 (1949).

This article contains a review of applications of the author's theory [J. Chem. Phys. 14, 180–201 (1946)] to transport processes in pure liquids by methods developed by Kirkwood, Buff, and Green [ibid. 17, 988–994 (1949)].

F. London (Durham, N. C.).

Gutkin, A. M. Concerning the theory of vapor. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 538–546 (1950). (Russian)

Pour étudier théoriquement les propriétés d'une vapeur saturée et surchauffée on considère fréquemment la vapeur comme un mélange d'un grand nombre de gaz. Le premier gaz du mélange est composé de molécules sans action les unes sur les autres; chaque particule du second gaz est formée d'une paire de molécules en interaction, mais sans action sur le reste des molécules, etc. La particule du $n$ème gaz est composée de n-molécules en interaction mais sans influence sur les autres. Pour chacun de ces gaz on calcule son énergie libre, et l'énergie libre de la vapeur, ainsi que le nombre des molécules de chaque gaz, sont calculées par la condition de minimum d'énergie libre dans un mélange gazeux. Frenkel [Kinetic Theory of Liquids, Oxford, 1946; ces Rev. 9, 168] considère chacun de ces gaz comme un gaz parfait. Band [J. Chem. Phys. 7, 324–326, 927–931 (1939)] et Tseng, Feng, Cheng, et Band [ibid. 8, 20–23 (1940)] introduisent une correction en calculant le volume propre de la vapeur. L'auteur reprend le calcul du volume propre de la molécule de vapeur en deux approximations successives et montre que, pour certaines valeurs de la densité de la vapeur, les termes de la seconde approximation peuvent devenir comparables à ceux de la première approximation. Il étudie le rôle des chocs élastiques et non élastiques. Il donne deux méthodes pour calculer la somme des états de la vapeur, l'une analogue à celle d'Ursell et une autre à celle de Frenkel. Il montre finalement que certaines considérations de Band paraissent improbables. *M. Kiveliovitch.*

Geilikman, B. T. On the statistical theory of phase transitions of the first order. Doklady Akad. Nauk SSSR (N.S.) 69, 329–332 (1949). (Russian)

The author discusses and criticizes Mayer's theory of condensation [cf., e.g., J. E. Mayer and M. G. Mayer, Statistical Mechanics, Wiley, New York, 1940, p. 277]. The theory leads to a limiting volume v_b such that $\partial P/\partial v = 0$ for $v < v_b$, corresponding to condensation. The author claims (no details given) that under the usual assumption of attractive molecular forces and positive cluster integrals ($b_i > 0$) one gets $\partial P/\partial v = 0$ also for $v \rightarrow v_b + 0$. Hence, this quantity is continuous, resulting in a third order rather than the experimentally found first order transition. This result is changed if repulsive forces and hence negative cluster integrals are admitted. *L. Tisza* (Cambridge, Mass.).

Geĭlikman, B. T. On the statistics of multicomponent systems. (Phase transitions of the second order.) Doklady Akad. Nauk SSSR (N.S.) **69**, 631–633 (1949). (Russian)

The first few irreducible cluster integrals of the Mayer theory are evaluated for special models: (1) hard sphere molecules, (2) point molecules with attractive forces, (3) attracting spheres, (4) attracting cube-like molecules. Conclusions are drawn concerning the nature of the resulting condensation in various temperature regions.

L. Tisza (Cambridge, Mass.).

Geĭlikman, B. T. On the statistics of condensed systems. Doklady Akad. Nauk SSSR (N.S.) **70**, 25–28 (1950). (Russian)

The author investigates the condensation of multicomponent gaseous systems [cf. K. Fuchs, Proc. Roy. Soc. London. Ser. A. **179**, 408–432 (1942); these Rev. **4**, 29]. He concludes that the condensation is of the second order with discontinuous values of $\partial P/\partial v$. *L. Tisza.*

Krastanow, L. Über einige Grundfragen bei den Kondensationsvorgängen in der Atmosphäre. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) **44**, 1–22 (1948). (Bulgarian. German summary)

Temperley, H. N. V. Statistical mechanics and the partition of numbers. I. The transition of liquid helium. Proc. Roy. Soc. London. Ser. A. **199**, 361–375 (1949).

The elementary theory of Bose-Einstein condensation is compared with some results obtained from the theory of partition of numbers. It is concluded that the elementary theory can be relied upon only at very high and at very low temperatures and also that there is a condensation phenomenon, but that it does not correctly describe the position of the transition temperature and the form of the specific heat anomaly of an ideal Bose-Einstein gas. The author does not give a correct calculation of these quantities nor does he see any reason to doubt either the reality of the condensation phenomenon or its relevance to the liquid helium problem.

F. London (Durham, N. C.).

Becker, R. Die Bose-Einstein-Kondensation als räumliches Phänomen. I. Z. Physik **128**, 120–132 (1950).

The author calculates the properties of an ideal Bose-Einstein gas in a gravitational field and shows that the Bose-Einstein condensation has in this case all features of an ordinary condensation with a phase separation in ordinary space. However the general conclusions drawn by the author concerning the nonexistence of a condensation in momentum space are not to the point, the latter having been discussed in the case of a Bose-Einstein liquid such as represented by a smoothed potential model.

F. London (Durham, N. C.).

Leibfried, Günther. Die Kondensation des idealen Bose-Gases als räumliches Phänomen. II. Z. Physik **128**, 133–143 (1950).

The distance correlations in an ideal Bose-Einstein gas [London, J. Chem. Phys. **11**, 203–213 (1943)] are interpreted as an expression of the density fluctuation of a vapor consisting of little droplets. The author comes to the result that the Bose droplets have an average diameter proportional to $n^{\frac{1}{3}}$ if n is the number of particles per droplet.

F. London (Durham, N. C.).

Müller, Henning. Zur Frage intermediärer Statistiken. Ann. Physik (6) **7**, 420–424 (1950).

The author discusses the possibility of applying Gentile's intermediate statistics to systems which consist of various distinguishable kinds of Bose and Fermi particles.

F. London (Durham, N. C.).

Gurov, K. P. On quantum hydrodynamics. II. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 279–285 (1950). (Russian)

This is a sequel to a previous paper of the author's [same Žurnal **18**, 110–125 (1948); these Rev. **10**, 92]. In this paper the author evaluates the coefficient of viscosity of a fluid, starting from the molecular model described in the previous paper. The viscosity is as usual a sum of two parts, a kinetic part depending only on the molecular velocity distribution functions, and a dynamical part depending explicitly also on the molecular interactions.

F. J. Dyson (Princeton, N. J.).

Bogoljubow, N. N. Zur Theorie der Superflüssigkeit. Sowjetwissenschaft **1948**, no. 1, 162–176 (1948).

Translated from Bull. Acad. Sci. URSS. Sér. Phys. [Izvestiya Akad. Nauk SSSR] **11**, 77–90 (1947); these Rev. **9**, 72.

Stupočenko, E. V. On the distribution of kinetic energy in reacting gaseous systems. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **19**, 493–501 (1949). (Russian)

L'auteur étudie la distribution de l'énergie cinétique des particules introduites dans un système gazeux par suite d'une réaction chimique ou d'un autre processus élémentaire. On suppose que la distribution des molécules du gaz est celle de Maxwell, que la vitesse de la réaction est lente, et que la densité du gaz n'est pas trop grande. Les particules de la réaction ont toutes la même masse et la même énergie ϵ_0. En utilisant l'équation de Boltzmann, l'auteur obtient une équation intégrale dont il donne la solution. Soit ϵ l'énergie après le choc, l'auteur trouve que la fonction de perturbation pour le domaine $\epsilon > \epsilon_0$ est représentée par une distribution de Maxwell de même température que celle du gaz, mais avec un nombre fictif des particules qu'il détermine. Et dans le domaine $\epsilon < \epsilon_0$ la courbe de Maxwell est déformée.

M. Kiveliovitch (Paris).

Klein, O. On the statistical derivation of the laws of chemical equilibrium. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 171–180 (1949).

Gibbs' well-known condition for equilibrium during a chemical reaction is derived in quantum statistics by use of the grand canonical distribution function. The relationship is then applied to equilibrium between atomic nuclei, nucleons, positive and negative electrons, neutrinos and light quanta (which is of importance in the thermodynamical theory of the origin of the elements). In this connection a generalization of a relationship of Tolman for temperature equilibrium in the general theory of relativity is given.

J. M. Luttinger (Madison, Wis.).

BIBLIOGRAPHICAL NOTES

Académie de la République Populaire Roumaine. Bulletin de la Section Scientifique.

This bulletin, formerly published by the Académie Roumaine, terminated in 1948 with volume 30. Its successor is: Academia Republicii Populare Române. Buletin Ştiinţific. A. Matematică, Fizică, Chimie, Geologie, Geografie, Biologie, Ştiinţe Tehnice şi Agricole. Volume 1 is dated 1949.

Acta Mathematica Academiae Scientiarum Hungaricae.

This journal, of which vol. 1, no. 1 is dated 1950, replaces Hungarica Acta Mathematica, four numbers of which appeared during the years 1946–49. The address is Acta Mathematica, Budapest 62, Postafiók 440.

Acta Univ. Szeged. Sect. Sci. Math.

Vol. 12 appeared in 1950, after vol. 13, and consists of two parts, A and B, instead of the usual four numbers. It carries the dedication: Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus.

Akademie der Wissenschaften und der Literatur in Mainz. Abhandlungen der mathematisch-naturwissenschaftlichen Klasse.

These Abhandlungen began with no. 1 in 1950.

Annali della Scuola Normale Superiore di Pisa.

The second series of this journal terminated with vol. 15 (1946) which was issued in 1950. The new (third) series will be published by the Scuola Normale Superiore, Pisa, instead of Zanichelli. Volumes 1 (1947), 2 (1948), and 3 (1949) of the new series are to be issued in the near future.

have been issued or

Archiv für mathematische Logik und Grundlagenforschung.

Vol. 1, no. 1, of this journal is dated September 1950. It is published by W. Kohlhammer Verlag, Stuttgart.

Australian Journal of Applied Science.

Volume 1, number 1, is dated March, 1950. This journal is published by the Commonwealth Scientific and Industrial Research Organization, 314 Albert Street, East Melbourne, C. 2, Victoria.

Bulletin de la Société des Mathématiciens et des Physiciens de la République Populaire de Macédoine.

The first volume of this bulletin, published in Skopje, appeared in 1950. The title also appears in Macedonian: Bilten na Društvoto na Matematičarite i Fizičarite od Narodna Republika Makedonija.

Cahiers Rhodaniens.

Number 1 appeared in 1949. This is a publication of the Institut de Mathématiques, Université de Lyon.

Canadian Journal of Research.

Section A of this journal has changed its title to Canadian Journal of Physics, starting with vol. 29, no. 1. Section B has changed to Canadian Journal of Chemistry, also starting with vol. 29, no. 1.

Japan Science Review. Series I. Engineering Sciences.

Volume 1, number 1, is dated March, 1949. The contents will include translations into European languages of scientific articles published in Japanese, in full as well as in condensed versions, and also new work. The journal is to be published quarterly by the Association for Science Documents Information, Tokyo Institute of Technology, Ōokayama, Meguro-ku, Tokyo, Japan.

Journal of the Institute of Polytechnics, Osaka City University. Series A, Mathematics.

Volume 1, number 1, is dated February, 1950.

La Ricerca. Rivista di Matematiche Pure ed Applicate.

Volume 1, number 1, is dated March, 1950. The journal is to be published quarterly by Conte, Via Mariano Semmola 45, Naples.

Nagoya Mathematical Journal.

Volume 1 is dated June, 1950. The journal is published by the Mathematical Institute of Nagoya University.

Schriftenreihe des Mathematischen Instituts der Universität Münster.

Heft 1 of this series is dated June 1948. The title of the series varies (Heft 3 has Schriften aus dem . . .).

Sowjetwissenschaft.

Volume 1 appeared in 1948, volume 2 in 1949, with four numbers in each volume, but three additional Beihefte in vol. 1. The journal is published by Verlag Kultur und Fortschritt, Berlin, and contains translations into German of various articles which have been published in Russian. Items of mathematical interest will be noted in these Reviews.

Studii şi Cercetări Matematice.

Vol. 1, no. 1, appeared in 1950; a volume will consist of two numbers. This is a publication of the Institututul de Matematică of the Academia Republicii Populare Române, Bucharest. It replaces two journals: Disquisitiones Mathematicae et Physicae, which terminated in 1948 with vol. 7; and Bulletin Mathématique de la Société Roumaine des Sciences, which terminated in 1947 with vol. 48. Papers will be in Romanian followed by a summary in Russian and one in French, English, German, or Italian.

Sūgaku [Mathematics].

Volume 1, number 1, appeared in April 1947. This journal is published by the Mathematical Society of Japan. Papers are in Japanese.

Summa Brasiliensis Mathematicae.

This journal, originally published by the Fundação Getúlio Vargas, is now being published by the Centro Brasileiro de Pesquisas Físicas, Avenida Pasteur 250, Rio de Janeiro.

Tensor.

A new series of this journal, published by The Tensor Society, Sapporo, Japan, was initiated in June, 1950 with vol. 1, no. 1. Papers of this number are in English. The old series with papers in Japanese terminated with no. 9 in 1949.

Ukrainskiĭ Matematičeskiĭ Žurnal.

This is a new publication of the Academy of Sciences of the Ukrainian SSR. Volume 1 appeared in 1949.

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